Basic Algebraic Operations

CHAPTER R

ALGEBRA is "generalized arithmetic." In arithmetic we add, subtract, multiply, and divide specific numbers. In algebra we use all that we know about arithmetic, but, in addition, we work with symbols that represent one or more numbers. In this chapter we review some important basic algebraic operations usually studied in earlier courses.

SECTIONS

R-1 Algebra and Real Numbers

- Sets
- The Set of Real Numbers
- The Real Number Line
- Addition and Multiplication of Real Numbers
- Further Operations and Properties

The numbers 14, -3, 0, $\frac{7}{5}$, $\sqrt{2}$, and $\sqrt[3]{6}$ are all examples of *real numbers*. Because the symbols we use in algebra often stand for real numbers, we will discuss important properties of the real number system. We first introduce some useful notions about sets.

Sets

Georg Cantor (1845–1918) developed a theory of sets as an outgrowth of his studies on infinity. His work has become a milestone in the development of mathematics.

Our use of the word "set" will not differ appreciably from the way it is used in everyday language. Words such as "set," "collection," "bunch," and "flock" all convey the same idea. Thus, we think of a **set** as a collection of objects with the important property that we can tell whether any given object is or is not in the set.

Each object in a set is called an **element,** or **member,** of the set. Symbolically,

$a \in A$	means	"a is an element of set A"	$3 \in \{1, 3, 5\}$
$a \notin A$	means	"a is not an element of set A"	$2 \notin \{1, 3, 5\}$

Capital letters are often used to represent sets and lowercase letters to represent elements of a set.

A set is **finite** if the number of elements in the set can be counted and **infinite** if there is no end in counting its elements. A set is **empty** if it contains no elements. The empty set is also called the **null** set and is denoted by $\varnothing$. It is important to observe that the empty set is *not* written as $\{\varnothing\}$.

A set is usually described in one of two ways—by **listing** the elements between braces, $\{\ \}$, or by enclosing within braces a **rule** that determines its elements. For example, if D is the set of all numbers x such that $x^2 = 4$, then using the listing method we write

$$D = \{-2, 2\} \quad \text{Listing method}$$

or, using the rule method we write

$$D = \{x \mid x^2 = 4\} \quad \text{Rule method}$$

The notation used in the rule method is sometimes called **set-builder notation;** the vertical bar $\mid$ represents "such that," and $\{x \mid x^2 = 4\}$ is read, "The set of all x such that $x^2 = 4$."

The letter x introduced in the rule method is a *variable.* In general, a **variable** is a symbol that is used as a placeholder for the elements of a set with two or more elements. This set is called the **replacement set** for the variable. A **constant,** on the other hand, is a symbol that names exactly one object. The symbol "8" is a constant, since it always names the number eight.

If each element of set A is also an element of set B, we say that A is a **subset** of set B, and we write

$$A \subset B \qquad \{1, 5\} \subset \{1, 3, 5\}$$

Note that the definition of a subset allows a set to be a subset of itself.

Since the empty set $\varnothing$ has no elements, every element of $\varnothing$ is also an element of any given set. Thus, the empty set is a subset of every set. For example,

$$\varnothing \subset \{1, 3, 5\} \qquad \text{and} \qquad \varnothing \subset \{2, 4, 6\}$$

If two sets A and B have exactly the same elements, the sets are said to be **equal,** and we write

$$A = B \qquad \{4, 2, 6\} = \{6, 4, 2\}$$

Notice that the order of listing elements in a set does not matter.

We can now begin our discussion of the real number system. Additional set concepts will be introduced as needed.

The Set of Real Numbers

The real number system is the number system you have used most of your life. Informally, a **real number** is any number that has a decimal representation. Table 1 describes the set of real numbers and some of its important subsets. Figure 1 on page 4 illustrates how these sets of numbers are related to each other.

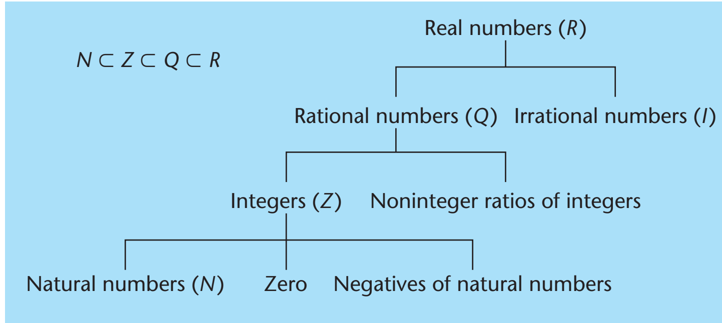
Table 1 The Set of Real Numbers

Symbol	Name	Description	Examples
N	Natural numbers	Counting numbers (also called positive integers)	1, 2, 3, . . .
Z	Integers	Natural numbers, their negatives, and 0 (also called whole numbers)	. . . , −2, −1, 0, 1, 2, . . .
Q	Rational numbers	Numbers that can be represented as a/b, where a and b are integers and $b \neq 0$; decimal representations are repeating or terminating	$-4, 0, 1, 25, \frac{-3}{5}, \frac{2}{3}, 3.67, -0.33\overline{3}$,* $5.2\overline{727}$
I	Irrational numbers	Numbers that can be represented as nonrepeating and nonterminating decimal numbers	$\sqrt{2}, \pi, \sqrt[3]{7}$, 1.414213 . . . ,† 2.71828182 . . .†
R	Real numbers	Rational numbers and irrational numbers	

*The overbar indicates that the number (or block of numbers) repeats indefinitely.

†Note that the ellipsis does *not* indicate that a number (or block of numbers) repeats indefinitely.

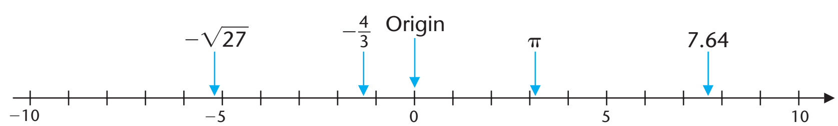
Figure 1
Real numbers and important subsets.



The Real Number Line

A one-to-one correspondence exists between the set of real numbers and the set of points on a line. That is, each real number corresponds to exactly one point, and each point to exactly one real number. A line with a real number associated with each point, and vice versa, as in Figure 2, is called a **real number line,** or simply a **real line.** Each number associated with a point is called the **coordinate** of the point. The point with coordinate 0 is called the **origin.** The arrow on the right end of the line indicates a positive direction. The coordinates of all points to the right of the origin are called **positive real numbers,** and those to the left of the origin are called **negative real numbers.** The real number 0 is neither positive nor negative.

Figure 2
A real number line.



Addition and Multiplication of Real Numbers

How do you add or multiply two real numbers that have nonrepeating and nonterminating decimal expansions? The answer to this difficult question relies on a solid understanding of the arithmetic of rational numbers. The **rational numbers** are numbers that can be written in the form a/b, where a and b are integers and $b \neq 0$ (see Table 1 on page 3). The numbers $7/5$ and $-2/3$ are rational, and any integer a is rational because it can be written in the form $a/1$. Two rational numbers a/b and c/d are **equal** if $ad = bc$; for example, $35/10 = 7/2$. Recall how the sum and product of rational numbers are defined.

DEFINITION 1 Addition and Multiplication of Rationals

For rational numbers a/b and c/d, where a, b, c, and d are integers and $b \neq 0$, $d \neq 0$:

Addition: $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Multiplication: $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Addition and multiplication of rational numbers are **commutative;** changing the order in which two numbers are added or multiplied does not change the result.

$$\frac{3}{2} + \frac{5}{7} = \frac{5}{7} + \frac{3}{2}$$ Addition is commutative.

$$\frac{3}{2} \cdot \frac{5}{7} = \frac{5}{7} \cdot \frac{3}{2}$$ Multiplication is commutative.

Addition and multiplication of rational numbers is also **associative;** changing the grouping of three numbers that are added or multiplied does not change the result:

$$\frac{3}{2} + \left(\frac{5}{7} + \frac{9}{4}\right) = \left(\frac{3}{2} + \frac{5}{7}\right) + \frac{9}{4}$$ Addition is associative.

$$\frac{3}{2} \cdot \left(\frac{5}{7} \cdot \frac{9}{4}\right) = \left(\frac{3}{2} \cdot \frac{5}{7}\right) \cdot \frac{9}{4}$$ Multiplication is associative.

Furthermore, the operations of addition and multiplication are related in that multiplication **distributes** over addition:

$$\frac{3}{2} \cdot \left(\frac{5}{7} + \frac{9}{4}\right) = \frac{3}{2} \cdot \frac{5}{7} + \frac{3}{2} \cdot \frac{9}{4}$$ Left distributive law

$$\left(\frac{5}{7} + \frac{9}{4}\right) \cdot \frac{3}{2} = \frac{5}{7} \cdot \frac{3}{2} + \frac{9}{4} \cdot \frac{3}{2}$$ Right distributive law

The rational number 0 is an **additive identity;** adding 0 to a number does not change it. The rational number 1 is a **multiplicative identity;** multiplying a number by 1 does not change it. Every rational number r has an **additive inverse,** denoted $-r$; the additive inverse of $4/5$ is $-4/5$, and the additive inverse of $-3/2$ is $3/2$. The sum of a number and its additive inverse is 0. Every *nonzero* rational number r has a **multiplicative inverse,** denoted r^{-1}; the multiplicative inverse of $4/5$ is $5/4$, and the multiplicative inverse of $-3/2$ is $-2/3$. The product of a number and its multiplicative inverse is 1. The rational number 0 has no multiplicative inverse.

EXAMPLE 1 Arithmetic of Rational Numbers

Perform the indicated operations.

(A) $\frac{1}{3} + \frac{6}{5}$ (B) $\frac{8}{3} \cdot \frac{5}{4}$

(C) $(-17/9)^{-1}$ (D) $(-6 + 9/2)^{-1}$

SOLUTION

(A) $\frac{1}{3} + \frac{6}{5} = \frac{5 + 18}{15} = \frac{23}{15}$

(B) $\frac{8}{3} \cdot \frac{5}{4} = \frac{40}{12} = \frac{10}{3}$ $\frac{40}{12} = \frac{10}{3}$ because $40 \cdot 3 = 12 \cdot 10$

(C) $(-17/9)^{-1} = -9/17$

(D) $(-6 + 9/2)^{-1} = \left(\frac{-6}{1} + \frac{9}{2}\right)^{-1} = \left(\frac{-12 + 9}{2}\right)^{-1} = \left(\frac{-3}{2}\right)^{-1} = \frac{-2}{3}$

MATCHED PROBLEM 1*

Perform the indicated operations.

(A) $-(5/2 + 7/3)$ (B) $-(8/17)^{-1}$

(C) $\frac{21}{20} \cdot \frac{15}{14}$ (D) $5 \cdot (1/2 + 1/3)$

Rational numbers have decimal expansions that are repeating or terminating. For example, by long division,

$$\frac{2}{3} = 0.66\overline{6}$$ The number 6 repeats indefinitely.

$$\frac{22}{7} = 3.\overline{142857}$$ The block 142857 repeats indefinitely.

$$\frac{13}{8} = 1.625$$ Terminating expansion

Conversely, any decimal expansion that is repeating or terminating represents a rational number (see Problems 69 and 70 in Exercise R-1).

The number $\sqrt{2}$ is *irrational* because it cannot be written in the form a/b, where a and b are integers, $b \neq 0$ (for an explanation, see Problem 83 in Section R-5 or the Chapter R Group Activity). Similarly, $\sqrt{3}$ is irrational. But $\sqrt{4}$, which is equal to 2, is a rational number. In fact, if n is a positive integer, then $\sqrt{n}$ is irrational unless n belongs to the sequence of perfect squares 1, 4, 9, 16, 25, . . . (see Problem 84 in Section R-5).

We now return to our original question: how do you add or multiply two real numbers that have nonrepeating and nonterminating decimal expansions? Although we will not give a detailed answer to this question, the key idea is that every real number can be approximated to any desired precision by rational numbers. For example, the irrational number

$$\sqrt{2} \approx 1.414\,213\,562\ldots$$

is approximated by the rational numbers

$$\frac{14}{10} = 1.4$$

$$\frac{141}{100} = 1.41$$

*Answers to matched problems in a given section are found near the end of the section, before the exercise set.

$$\frac{1{,}414}{1{,}000} = 1.414$$

$$\frac{14{,}142}{10{,}000} = 1.4142$$

$$\frac{141{,}421}{100{,}000} = 1.41421$$

$$\vdots$$

Using the idea of approximation by rational numbers, we can extend the definitions of rational number operations to include real number operations. The following box summarizes the basic properties of real number operations.

› BASIC PROPERTIES OF THE SET OF REAL NUMBERS

Let R be the set of real numbers, and let x, y, and z be arbitrary elements of R.

Addition Properties

Closure: $x + y$ is a unique element in R.

Associative: $(x + y) + z = x + (y + z)$

Commutative: $x + y = y + x$

Identity: 0 is the additive identity; that is, $0 + x = x + 0 = x$ for all x in R, and 0 is the only element in R with this property.

Inverse: For each x in R, $-x$ is its unique additive inverse; that is, $x + (-x) = (-x) + x = 0$, and $-x$ is the only element in R relative to x with this property.

Multiplication Properties

Closure: xy is a unique element in R.

Associative: $(xy)z = x(yz)$

Commutative: $xy = yx$

Identity: 1 is the multiplicative identity; that is, for all x in R, $(1)x = x(1) = x$, and 1 is the only element in R with this property.

Inverse: For each x in R, $x \neq 0$, x^{-1} is its unique multiplicative inverse; that is, $xx^{-1} = x^{-1}x = 1$, and x^{-1} is the only element in R relative to x with this property.

Combined Property

Distributive: $x(y + z) = xy + xz \qquad (x + y)z = xz + yz$

EXAMPLE 2 Using Real Number Properties

Which real number property justifies the indicated statement?

Statement	**Property Illustrated**
(A) $(7x)y = 7(xy)$	Associative $(\cdot)$
(B) $a(b + c) = (b + c)a$	Commutative $(\cdot)$
(C) $(2x + 3y) + 5y = 2x + (3y + 5y)$	Associative $(+)$
(D) $(x + y)(a + b) = (x + y)a + (x + y)b$	Distributive
(E) If $a + b = 0$, then $b = -a$.	Inverse $(+)$

MATCHED PROBLEM 2

Which real number property justifies the indicated statement?

(A) $4 + (2 + x) = (4 + 2) + x$

(B) $(a + b) + c = c + (a + b)$

(C) $3x + 7x = (3 + 7)x$

(D) $(2x + 3y) + 0 = 2x + 3y$

(E) If $ab = 1$, then $b = 1/a$.

Further Operations and Properties

Subtraction of real numbers can be defined in terms of addition and the additive inverse. If a and b are real numbers, then $a - b$ is defined to be $a + (-b)$. Similarly, division can be defined in terms of multiplication and the multiplicative inverse. If a and b are real numbers and $b \neq 0$, then $a \div b$ (also denoted a/b) is defined to be $a \cdot b^{-1}$.

DEFINITION 2 Subtraction and Division of Real Numbers

For all real numbers a and b:

Subtraction: $a - b = a + (-b)$ $\quad 5 - 3 = 5 + (-3) = 2$

Division: $a \div b = a \cdot b^{-1} \quad b \neq 0$ $\quad 3 \div 2 = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2} = 1.5$

It is important to remember that

Division by 0 is never allowed.

››› EXPLORE-DISCUSS 1

(A) Give an example that shows that subtraction of real numbers is not commutative.

(B) Give an example that shows that division of real numbers is not commutative.

The basic properties of the set of real numbers, together with the definitions of subtraction and division, imply the following properties of negatives and zero.

› THEOREM 1 Properties of Negatives

For all real numbers a and b:

1. $-(-a) = a$
2. $(-a)b = -(ab) = a(-b) = -ab$
3. $(-a)(-b) = ab$
4. $(-1)a = -a$
5. $\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b} \qquad b \neq 0$
6. $\frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b} = \frac{a}{b} \qquad b \neq 0$

› THEOREM 2 Zero Properties

For all real numbers a and b:

1. $a \cdot 0 = 0 \cdot a = 0$
2. $ab = 0$ if and only if* $a = 0$ or $b = 0$ or both

Note that if $b \neq 0$, then $\frac{0}{b} = 0 \cdot b^{-1} = 0$ by Theorem 2. In particular, $\frac{0}{3} = 0$; but the expressions $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.

*Given statements P and Q, "P if and only if Q" stands for both "if P, then Q" and "if Q, then P."

EXAMPLE 3 Using Negative and Zero Properties

Which real number property or definition justifies each statement?

Statement	Property or Definition Illustrated
(A) $3 - (-2) = 3 + [-(-2)] = 5$	Subtraction (Definition 1 and Theorem 1, part 1)
(B) $-(-2) = 2$	Negatives (Theorem 1, part 1)
(C) $-\frac{-3}{2} = \frac{3}{2}$	Negatives (Theorem 1, part 6)
(D) $\frac{5}{-2} = -\frac{5}{2}$	Negatives (Theorem 1, part 5)
(E) If $(x - 3)(x + 5) = 0$, then either $x - 3 = 0$ or $x + 5 = 0$.	Zero (Theorem 2, part 2)

MATCHED PROBLEM 3

Which real number property or definition justifies each statement?

(A) $\frac{3}{5} = 3\left(\frac{1}{5}\right)$ (B) $(-5)(2) = -(5 \cdot 2)$ (C) $(-1)3 = -3$

(D) $\frac{-7}{9} = -\frac{7}{9}$ (E) If $x + 5 = 0$, then $(x - 3)(x + 5) = 0$.

››› EXPLORE-DISCUSS 2

A set of numbers is **closed** under an operation if performing the operation on numbers of the set always produces another number in the set. For example, the set of odd integers is closed under multiplication, but is not closed under addition.

(A) Give an example that shows that the set of irrational numbers is not closed under addition.

(B) Explain why the set of irrational numbers is closed under taking multiplicative inverses.

If a and b are real numbers, $b \neq 0$, the quotient $a \div b$, when written in the form a/b, is called a **fraction.** The number a is the **numerator,** and b is the **denominator.** It can be shown that fractions satisfy the following properties. (Note that some of these properties, under the restriction that numerators and denominators be integers, were used earlier to define arithmetic operations on the rationals.)

THEOREM 3 Fraction Properties

For all real numbers a, b, c, d, and k (division by 0 excluded):

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$

   $\frac{4}{6} = \frac{6}{9}$ since $4 \cdot 9 = 6 \cdot 6$

2. $\frac{ka}{kb} = \frac{a}{b}$

   $\frac{7 \cdot 3}{7 \cdot 5} = \frac{3}{5}$

3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

   $\frac{3}{5} \cdot \frac{7}{8} = \frac{3 \cdot 7}{5 \cdot 8} = \frac{21}{40}$

4. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

   $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$

5. $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

   $\frac{3}{6} + \frac{4}{6} = \frac{3 + 4}{6} = \frac{7}{6}$

6. $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$

   $\frac{7}{8} - \frac{2}{8} = \frac{7 - 2}{8} = \frac{5}{8}$

7. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

   $\frac{2}{3} + \frac{1}{5} = \frac{2 \cdot 5 + 3 \cdot 1}{3 \cdot 5} = \frac{13}{15}$

ANSWERS TO MATCHED PROBLEMS

1. (A) $-29/6$ (B) $-17/8$ (C) $9/8$ (D) $25/6$

2. (A) Associative (+) (B) Commutative (+) (C) Distributive
(D) Identity (+) (E) Inverse (·)

3. (A) Division (Definition 1) (B) Negatives (Theorem 1, part 2)
(C) Negatives (Theorem 1, part 4) (D) Negatives (Theorem 1, part 5)
(E) Zero (Theorem 2, part 1)

R-1 Exercises

In Problems 1–12, indicate true (T) or false (F).

1. $4 \in \{3, 4, 5\}$

2. $6 \in \{2, 4, 6\}$

3. $3 \notin \{3, 4, 5\}$

4. $7 \notin \{2, 4, 6\}$

5. $\{1, 2\} \subset \{1, 3, 5\}$

6. $\{2, 6\} \subset \{2, 4, 6\}$

7. $\{7, 3, 5\} \subset \{3, 5, 7\}$

8. $\{7, 3, 5\} = \{3, 5, 7\}$

9. $\varnothing \in \{A, B, C\}$

10. $\varnothing \subset \{A, B, C, D\}$

11. $\{A, L, G, E, B, R, A\} = \{G, A, R, B, L, E\}$

12. $\{A, B, C, D, E\} \subset \{A, C, E\}$

In Problems 13–20, perform the indicated operations, if defined, and express the result as an integer.

13. $\frac{4}{9} \cdot \frac{9}{4}$

14. $\left(-\frac{3}{5}\right)\left(-\frac{5}{3}\right)$

15. $100 \div 0$

16. $0 \div 0$

17. $0 \div (9 \cdot 8)$

18. $\frac{4}{7} \div \left(3 - \frac{6}{2}\right)$

19. $0 \cdot \left(100 + \frac{1}{100}\right)$

20. $[(0 + 5) \div 20]^{-1}$

In Problems 21–26, perform the indicated operations and express the result in the form a/b, where a and b are integers.

21. $\frac{4}{9} + \frac{12}{5}$

22. $\frac{17}{8} \cdot \frac{2}{7}$

23. $-\left(\frac{1}{100} + \frac{4}{25}\right)$

24. $\left(\frac{5}{2} \cdot \frac{14}{25}\right)^{-1}$

25. $\left(\frac{3}{8}\right)^{-1} + 2^{-1}$

26. $-(1 + 3^{-1})$

In Problems 27–38, each statement illustrates the use of one of the following properties or definitions. Indicate which one.

Commutative (+)	*Inverse (+)*
Commutative (·)	*Inverse (·)*
Associative (+)	*Subtraction*
Associative (·)	*Division*
Distributive	*Negatives (Theorem 1)*
Identity (+)	*Zero (Theorem 2)*
Identity (·)	

27. $x + ym = x + my$

28. $7(3m) = (7 \cdot 3)m$

29. $7u + 9u = (7 + 9)u$

30. $-\frac{u}{-v} = \frac{u}{v}$

31. $(-2)(\frac{1}{-2}) = 1$

32. $8 - 12 = 8 + (-12)$

33. $w + (-w) = 0$

34. $5 \div (-6) = 5(\frac{1}{-6})$

35. $3(xy + z) + 0 = 3(xy + z)$

36. $ab(c + d) = abc + abd$

37. $\frac{-x}{-y} = \frac{x}{y}$

38. $(x + y) \cdot 0 = 0$

Write each set in Problems 39–44 using the listing method; that is, list the elements between braces. If the set is empty, write ∅.

39. $\{x \mid x$ is an even integer between -3 and $5\}$

40. $\{x \mid x$ is an odd integer between -4 and $6\}$

41. $\{x \mid x$ is a letter in "status"$\}$

42. $\{x \mid x$ is a letter in "consensus"$\}$

43. $\{x \mid x$ is a month starting with B$\}$

44. $\{x \mid x$ is a month with 32 days$\}$

45. The set $S_1 = \{a\}$ has only two subsets, S_1 and ∅. How many subsets does each of the following sets have?
(A) $S_2 = \{a, b\}$
(B) $S_3 = \{a, b, c\}$
(C) $S_4 = \{a, b, c, d\}$

46. Based on the results in Problem 45, how many subsets do you think a set with n elements will have?

47. If $ab = 0$, does either a or b have to be 0?

48. If $ab = 1$, does either a or b have to be 1?

49. Indicate which of the following are true:
(A) All natural numbers are integers.
(B) All real numbers are irrational.
(C) All rational numbers are real numbers.

50. Indicate which of the following are true:
(A) All integers are natural numbers.
(B) All rational numbers are real numbers.
(C) All natural numbers are rational numbers.

51. Give an example of a rational number that is not an integer.

52. Give an example of a real number that is not a rational number.

In Problems 53 and 54, list the subset of S consisting of (A) natural numbers, (B) integers, (C) rational numbers, and (D) irrational numbers.

53. $S = \{-3, -\frac{2}{3}, 0, 1, \sqrt{3}, \frac{9}{5}, \sqrt{144}\}$

54. $S = \{-\sqrt{5}, -1, -\frac{1}{2}, 2, \sqrt{7}, 6, \sqrt{625/9}, \pi\}$

In Problems 55 and 56, use a calculator to express each number in decimal form. Classify each decimal number as terminating, repeating, or nonrepeating and nonterminating. Identify the pattern of repeated digits in any repeating decimal numbers.*

55. (A) $\frac{8}{9}$ (B) $\frac{3}{11}$ (C) $\sqrt{5}$ (D) $\frac{11}{8}$

56. (A) $\frac{13}{6}$ (B) $\sqrt{21}$ (C) $\frac{7}{16}$ (D) $\frac{29}{111}$

57. Indicate true (T) or false (F), and for each false statement find real number replacements for a and b that will provide a counterexample. For all real numbers a and b:
(A) $a + b = b + a$
(B) $a - b = b - a$
(C) $ab = ba$
(D) $a \div b = b \div a$

58. Indicate true (T) or false (F), and for each false statement find real number replacements for a, b, and c that will provide a counterexample. For all real numbers a, b, and c:
(A) $(a + b) + c = a + (b + c)$
(B) $(a - b) - c = a - (b - c)$
(C) $a(bc) = (ab)c$
(D) $(a \div b) \div c = a \div (b \div c)$

*Later in the book you will encounter optional exercises that require a graphing calculator. If you have such a calculator, you can certainly use it here. Otherwise, any scientific calculator will be sufficient for the problems in this chapter.

In Problems 59–66, indicate true (T) or false (F), and for each false statement give a specific counterexample.

59. The difference of any two natural numbers is a natural number.

60. The quotient of any two nonzero integers is an integer.

61. The sum of any two rational numbers is a rational number.

62. The sum of any two irrational numbers is an irrational number.

63. The product of any two irrational numbers is an irrational number.

64. The product of any two rational numbers is a rational number.

65. The multiplicative inverse of any irrational number is an irrational number.

66. The multiplicative inverse of any nonzero rational number is a rational number.

67. If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, find:
(A) $\{x \mid x \in A \text{ or } x \in B\}$
(B) $\{x \mid x \in A \text{ and } x \in B\}$

68. If $F = \{-2, 0, 2\}$ and $G = \{-1, 0, 1, 2\}$, find:
(A) $\{x \mid x \in F \text{ or } x \in G\}$
(B) $\{x \mid x \in F \text{ and } x \in G\}$

69. If $c = 0.151515\ldots$, then $100c = 15.1515\ldots$ and

$$100c - c = 15.1515\ldots - 0.151515\ldots$$
$$99c = 15$$
$$c = \tfrac{15}{99} = \tfrac{5}{33}$$

Proceeding similarly, convert the repeating decimal 0.090909 . . . into a fraction. (All repeating decimals are rational numbers, and all rational numbers have repeating decimal representations.)

70. Repeat Problem 69 for 0.181818. . . .

71. To see how the distributive property is behind the mechanics of long multiplication, compute each of the following and compare:

Long Multiplication	Use of the Distributive Property
$\begin{array}{r} 23 \\ \times\, 12 \\ \hline \end{array}$	$23 \cdot 12 = 23(2 + 10) = 23 \cdot 2 + 23 \cdot 10 =$

72. For a and b real numbers, justify each step using a property in this section.

Statement	Reason
1. $(a + b) + (-a) = (-a) + (a + b)$	**1.**
2. $= [(-a) + a] + b$	**2.**
3. $= 0 + b$	**3.**
4. $= b$	**4.**

R-2 Exponents

- Integer Exponents
- Roots of Real Numbers
- Rational Exponents
- Scientific Notation

The French philosopher/mathematician René Descartes (1596–1650) is generally credited with the introduction of the very useful exponent notation "x^n." This notation as well as other improvements in algebra may be found in his *Geometry,* published in 1637.

If n is a natural number, x^n denotes the product of n factors, each equal to x. In this section the meaning of x^n will be expanded to allow the exponent n to be any rational number. Each of the following expressions will then represent a unique real number:

$$7^5 \qquad 5^{-4} \qquad 3.14^0 \qquad 6^{1/2} \qquad 14^{-5/3}$$

Integer Exponents

Definition 1 generalizes exponent notation to include 0 and negative integer exponents.

DEFINITION 1 a^n, n an integer and a a real number

1. For n a positive integer:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a} \qquad 3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

2. For $n = 0$:

$$a^0 = 1 \qquad a \neq 0 \qquad 132^0 = 1$$

0^0 is not defined

3. For n a negative integer:

$$a^n = \frac{1}{a^{-n}} \qquad a \neq 0 \qquad 7^{-3} \boxed{= \frac{1}{7^{-(-3)}}}^* = \frac{1}{7^3}$$

Note: It can be shown that for any $a \neq 0$ and for *all* integers n

$$a^{-n} = \frac{1}{a^n} \qquad a^{-5} = \frac{1}{a^5} \qquad a^{-(-3)} = \frac{1}{a^{-3}}$$

EXAMPLE 1 Using the Definition of Integer Exponents

Write each part as a decimal fraction or using positive exponents. Assume all variables represent nonzero real numbers.

(A) $(u^3v^2)^0 = 1$ (B) $10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = 0.001$

(C) $x^{-8} = \frac{1}{x^8}$ (D) $\frac{x^{-3}}{y^{-5}} \boxed{= \frac{x^{-3}}{1} \cdot \frac{1}{y^{-5}} = \frac{1}{x^3} \cdot \frac{y^5}{1}} = \frac{y^5}{x^3}$

MATCHED PROBLEM 1

Write parts (A)–(D) as decimal fractions and parts (E) and (F) with positive exponents. Assume all variables represent nonzero real numbers.

(A) 636^0 (B) $(x^2)^0$ (C) 10^{-5}

(D) $\frac{1}{10^{-3}}$ (E) $\frac{1}{x^{-4}}$ (F) $\frac{u^{-7}}{v^{-3}}$

*Throughout the book, dashed boxes—called **think boxes**—are used to represent steps that may be performed mentally.

The basic properties of integer exponents are summarized in Theorem 1. The proof of this theorem involves *mathematical induction,* which is discussed in Chapter 8.

THEOREM 1 Properties of Integer Exponents

For n and m integers and a and b real numbers:

1. $a^m a^n = a^{m+n}$ $\qquad a^5a^{-7} = a^{5+(-7)} = a^{-2}$
2. $(a^n)^m = a^{mn}$ $\qquad (a^3)^{-2} = a^{(-2)3} = a^{-6}$
3. $(ab)^m = a^m b^m$ $\qquad (ab)^3 = a^3b^3$
4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} \quad b \neq 0$ $\qquad \left(\dfrac{a}{b}\right)^4 = \dfrac{a^4}{b^4}$
5. $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n} \\ \dfrac{1}{a^{n-m}} \end{cases} \quad a \neq 0$ $\qquad \dfrac{a^3}{a^{-2}} = a^{3-(-2)} = a^5$; $\quad \dfrac{a^3}{a^{-2}} = \dfrac{1}{a^{-2-3}} = \dfrac{1}{a^{-5}}$

EXPLORE-DISCUSS 1

Property 1 in Theorem 1 can be expressed verbally as follows:

To find the product of two exponential forms with the same base, add the exponents and use the same base.

Express the other properties in Theorem 1 verbally. Decide which you find easier to remember, a formula or a verbal description.

EXAMPLE 2 Using Exponent Properties

Simplify using exponent properties, and express answers using positive exponents only.*

(A) $(3a^5)(2a^{-3}) = (3 \cdot 2)(a^5a^{-3}) = 6a^2$

(B) $\dfrac{6x^{-2}}{8x^{-5}} = \dfrac{3x^{-2-(-5)}}{4} = \dfrac{3x^3}{4}$

*By "simplify" we mean eliminate common factors from numerators and denominators and reduce to a minimum the number of times a given constant or variable appears in an expression. We ask that answers be expressed using positive exponents only in order to have a definite form for an answer. Later (in this section and elsewhere) we will encounter situations where we will want negative exponents in a final answer.

(C) $-4y^3 - (-4y)^3 = -4y^3 - (-4)^3y^3$ $= -4y^3 - (-64)y^3$

$= -4y^3 + 64y^3 = 60y^3$

MATCHED PROBLEM 2

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(5x^{-3})(3x^4)$ (B) $\frac{9y^{-7}}{6y^{-4}}$ (C) $2x^4 - (-2x)^4$

››› CAUTION ›››

Be careful when using the relationship $a^{-n} = \frac{1}{a^n}$:

$ab^{-1} \neq \frac{1}{ab}$ $ab^{-1} = \frac{a}{b}$ and $(ab)^{-1} = \frac{1}{ab}$

$\frac{1}{a+b} \neq a^{-1} + b^{-1}$ $\frac{1}{a+b} = (a+b)^{-1}$ and $\frac{1}{a} + \frac{1}{b} = a^{-1} + b^{-1}$

Do not confuse properties 1 and 2 in Theorem 1:

$a^3a^4 \neq a^{3\cdot 4}$ $a^3a^4 = a^{3+4} = a^7$ property 1, Theorem 1

$(a^3)^4 \neq a^{3+4}$ $(a^3)^4 = a^{3\cdot 4} = a^{12}$ property 2, Theorem 1

From the definition of negative exponents and the five properties of exponents, we can easily establish the following properties, which are used very frequently when dealing with exponent forms.

› THEOREM 2 Further Exponent Properties

For a and b any real numbers and m, n, and p any integers (division by 0 excluded):

1. $(a^mb^n)^p = a^{pm}b^{pn}$ **2.** $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{pm}}{b^{pn}}$

3. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$ **4.** $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

PROOF We prove properties 1 and 4 in Theorem 2 and leave the proofs of 2 and 3 to you.

1. $(a^m b^n)^p = (a^m)^p (b^n)^p$ property 3, Theorem 1

$= a^{pm} b^{pn}$ property 2, Theorem 1

4. $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{a^{-n}}{b^{-n}}$ property 4, Theorem 1

$= \dfrac{b^n}{a^n}$ property 3, Theorem 2

$= \left(\dfrac{b}{a}\right)^n$ property 4, Theorem 1

EXAMPLE 3 Using Exponent Properties

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(2a^{-3}b^2)^{-2} = 2^{-2}a^6b^{-4} = \dfrac{a^6}{4b^4}$

(B) $\left(\dfrac{a^3}{b^5}\right)^{-2} = \dfrac{a^{-6}}{b^{-10}} = \dfrac{b^{10}}{a^6}$ or $\left(\dfrac{a^3}{b^5}\right)^{-2} = \left(\dfrac{b^5}{a^3}\right)^2 = \dfrac{b^{10}}{a^6}$

(C) $\dfrac{4x^{-3}y^{-5}}{6x^{-4}y^3} = \dfrac{2x^{-3-(-4)}}{3y^{3-(-5)}} = \dfrac{2x}{3y^8}$

(D) $\left(\dfrac{m^{-3}m^3}{n^{-2}}\right)^{-2} = \left(\dfrac{m^{-3+3}}{n^{-2}}\right)^{-2} = \left(\dfrac{m^0}{n^{-2}}\right)^{-2} = \left(\dfrac{1}{n^{-2}}\right)^{-2} = \dfrac{1}{n^4}$

(E) $(x + y)^{-3} = \dfrac{1}{(x + y)^3}$

MATCHED PROBLEM 3

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(3x^4y^{-3})^{-2}$ (B) $\left(\dfrac{x^2}{y^4}\right)^{-3}$ (C) $\dfrac{6m^{-2}n^3}{15m^{-1}n^{-2}}$

(D) $\left(\dfrac{x^{-3}}{y^4y^{-4}}\right)^{-3}$ (E) $\dfrac{1}{(a - b)^{-2}}$

In simplifying exponent forms there is often more than one sequence of steps that will lead to the same result (see Example 3B). Use whichever sequence of steps makes sense to you.

Roots of Real Numbers

Perhaps you recall that a **square root** of a number b is a number c such that $c^2 = b$, and a **cube root** of a number b is a number d such that $d^3 = b$.

What are the square roots of 9?

$$3 \text{ is a square root of } 9, \text{ since } 3^2 = 9.$$
$$-3 \text{ is a square root of } 9, \text{ since } (-3)^2 = 9.$$

Thus, 9 has two real square roots, one the negative of the other.

What are the cube roots of 8?

$$2 \text{ is a cube root of } 8, \text{ since } 2^3 = 8.$$

And 2 is the only real number with this property. In general:

DEFINITION 2 Definition of an *n*th Root

For a natural number n and a and b real numbers:

a is an ***n*th root** of b if $a^n = b$ 3 is a fourth root of 81, since $3^4 = 81$.

How many real square roots of 4 exist? Of 5? Of −9? How many real fourth roots of 5 exist? Of −5? How many real cube roots of 27 are there? Of −27? The following important theorem (which we state without proof) answers these questions.

THEOREM 3 Number of Real *n*th Roots of a Real Number *b**

	***n* even**	***n* odd**
b positive	Two real nth roots −3 and 3 are both fourth roots of 81.	One real nth root 2 is the only real cube root of 8.
b negative	No real nth root −9 has no real square roots.	One real nth root −2 is the only real cube root of −8.

*In this section we limit our discussion to real roots of real numbers. After the real numbers are extended to the complex numbers (see Section 1-4), additional roots may be considered. For example, it turns out that 1 has three cube roots: in addition to the real number 1, there are two other cube roots of 1 in the complex number system.

Thus, 4 and 5 have two real square roots each, and -9 has none. There are two real fourth roots of 5 and none for -5. And 27 and -27 have one real cube root each. What symbols do we use to represent these roots? We turn to this question now.

› Rational Exponents

If all exponent properties are to continue to hold even if some of the exponents are rational numbers, then

$$(5^{1/3})^3 = 5^{3/3} = 5 \qquad \text{and} \qquad (7^{1/2})^2 = 7^{2/2} = 7$$

Since Theorem 3 states that the number 5 has one real cube root, it seems reasonable to use the symbol $5^{1/3}$ to represent this root. On the other hand, Theorem 3 states that 7 has two real square roots. Which real square root of 7 does $7^{1/2}$ represent? We answer this question in the following definition.

› DEFINITION 3 $b^{1/n}$, Principal nth Root

For n a natural number and b a real number,

$b^{1/n}$ is the **principal nth root of b**

defined as follows:

1. If n is even and b is positive, then $b^{1/n}$ represents the positive nth root of b.

   $16^{1/2} = 4$ not -4 and 4.

   $-16^{1/2} = -4$ $-16^{1/2}$ and $(-16)^{1/2}$ are not the same.

2. If n is even and b is negative, then $b^{1/n}$ does not represent a real number. (More will be said about this case later.)

   $(-16)^{1/2}$ is not real.

3. If n is odd, then $b^{1/n}$ represents the real nth root of b (there is only one).

   $32^{1/5} = 2$ $(-32)^{1/5} = -2$

4. $0^{1/n} = 0$ $0^{1/9} = 0$ $0^{1/6} = 0$

EXAMPLE 4 Principal nth Roots

(A) $9^{1/2} = 3$

(B) $-9^{1/2} = -3$ Compare parts (B) and (C).

(C) $(-9)^{1/2}$ is not a real number.

(D) $27^{1/3} = 3$

(E) $(-27)^{1/3} = -3$

(F) $0^{1/7} = 0$

MATCHED PROBLEM 4

Find each of the following:

(A) $4^{1/2}$ (B) $-4^{1/2}$ (C) $(-4)^{1/2}$

(D) $8^{1/3}$ (E) $(-8)^{1/3}$ (F) $0^{1/8}$

How should a symbol such as $7^{2/3}$ be defined? If the properties of exponents are to hold for rational exponents, then $7^{2/3} = (7^{1/3})^2$; that is, $7^{2/3}$ must represent the square of the cube root of 7. This leads to the following general definition:

DEFINITION 4 $b^{m/n}$ and $b^{-m/n}$, Rational Number Exponent

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$b^{m/n} = (b^{1/n})^m \qquad \text{and} \qquad b^{-m/n} = \frac{1}{b^{m/n}}$$

$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$ $\quad 4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{8}$ $\quad (-4)^{3/2}$ is not real

$(-32)^{3/5} = [(-32)^{1/5}]^3 = (-2)^3 = -8$

We have now discussed $b^{m/n}$ for all rational numbers m/n and real numbers b. It can be shown, though we will not do so, that all five properties of exponents listed in Theorem 1 continue to hold for rational exponents as long as we avoid even roots of negative numbers. With the latter restriction in effect, the following useful relationship is an immediate consequence of the exponent properties:

THEOREM 4 Rational Exponent Property

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$b^{m/n} = \begin{cases} (b^{1/n})^m \\ (b^m)^{1/n} \end{cases} \qquad 8^{2/3} = \begin{cases} (8^{1/3})^2 \\ (8^2)^{1/3} \end{cases}$$

››› EXPLORE-DISCUSS 2

Find the contradiction in the following chain of equations:

$$-1 = (-1)^{2/2} = [(-1)^2]^{1/2} = 1^{1/2} = 1 \qquad (1)$$

Where did we try to use Theorem 4? Why was this not correct?

The three exponential forms in Theorem 4 are equal as long as only real numbers are involved. But if b is negative and n is even, then $b^{1/n}$ is not a real number and Theorem 4 does not necessarily hold, as illustrated in Explore-Discuss 2. One way to avoid this difficulty is to assume that m and n have no common factors.

EXAMPLE 5 Using Rational Exponents

Simplify, and express answers using positive exponents only. All letters represent positive real numbers.

(A) $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$ or $8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$

(B) $(-8)^{5/3} = [(-8)^{1/3}]^5 = (-2)^5 = -32$

(C) $(3x^{1/3})(2x^{1/2}) = 6x^{1/3+1/2} = 6x^{5/6}$

(D) $\left(\dfrac{4x^{1/3}}{x^{1/2}}\right)^{1/2} = \dfrac{4^{1/2}x^{1/6}}{x^{1/4}} = \dfrac{2}{x^{1/4-1/6}} = \dfrac{2}{x^{1/12}}$

(E) $(u^{1/2} - 2v^{1/2})(3u^{1/2} + v^{1/2}) = 3u - 5u^{1/2}v^{1/2} - 2v$

MATCHED PROBLEM 5

Simplify, and express answers using positive exponents only. All letters represent positive real numbers.

(A) $9^{3/2}$ (B) $(-27)^{4/3}$ (C) $(5y^{3/4})(2y^{1/3})$ (D) $(2x^{-3/4}y^{1/4})^4$

(E) $\left(\dfrac{8x^{1/2}}{x^{2/3}}\right)^{1/3}$ (F) $(2x^{1/2} + y^{1/2})(x^{1/2} - 3y^{1/2})$

› Scientific Notation

Scientific work often involves the use of very large numbers or very small numbers. For example, the average cell contains about 200,000,000,000,000 molecules, and the diameter of an electron is about 0.000 000 000 0004 centimeter. It is generally troublesome to write and work with numbers of this type in standard decimal form. The two numbers written here cannot even be entered into most calculators as they are written. However, each can be expressed as the product of a number between 1 and 10 and an integer power of 10:

$$200{,}000{,}000{,}000{,}000 = 2 \times 10^{14}$$
$$0.000\ 000\ 000\ 0004 = 4 \times 10^{-13}$$

In fact any positive number written in decimal form can be expressed in **scientific notation,** that is, in the form

$$a \times 10^n \qquad 1 \le a < 10,\ n \text{ an integer, } a \text{ in decimal form}$$

EXAMPLE 6

Scientific Notation

Each number is written in scientific notation:

$$7 = 7 \times 10^0 \qquad 0.5 = 5 \times 10^{-1}$$
$$720 = 7.2 \times 10^2 \qquad 0.08 = 8 \times 10^{-2}$$
$$6{,}430 = 6.43 \times 10^3 \qquad 0.000\ 32 = 3.2 \times 10^{-4}$$
$$5{,}350{,}000 = 5.35 \times 10^6 \qquad 0.000\ 000\ 0738 = 7.38 \times 10^{-8}$$

Can you discover a rule relating the number of decimal places the decimal point is moved to the power of 10 that is used?

$$7{,}320{,}000 = 7.320\ 000. \times 10^6 = 7.32 \times 10^6$$

6 places left
Positive exponent

$$0.000\ 000\ 54 = 0.000\ 000\ 5.4 \times 10^{-7} = 5.4 \times 10^{-7}$$

7 places right
Negative exponent

MATCHED PROBLEM 6

(A) Write each number in scientific notation: 430; 23,000; 345,000,000; 0.3; 0.0031; 0.000 000 683

(B) Write in standard decimal form: 4×10^3; 5.3×10^5; 2.53×10^{-2}; 7.42×10^{-6}

Most calculators express very large and very small numbers in scientific notation. (Later in the book you will encounter optional exercises that require a graphing calculator. If you have such a calculator, you can certainly use it here. Otherwise, any scientific calculator will be sufficient for the problems in this chapter.) Consult the manual for your calculator to see how numbers in scientific notation are entered in your calculator. Some common methods for displaying scientific notation on a calculator are shown here.

Number Represented	Typical Scientific Calculator Display	Typical Graphing Calculator Display
$5.427\ 493 \times 10^{-17}$	5.427493 – 17	5.427493E–17
$2.359\ 779 \times 10^{12}$	2.359779 12	2.359779E12

EXAMPLE 7

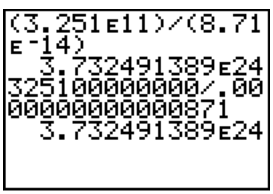
Using Scientific Notation on a Calculator

Write each number in scientific notation; then carry out the computations using your calculator. (Refer to the user's manual accompanying your calculator for the procedure.) Express the answer to three significant digits* in scientific notation.

$$\frac{325{,}100{,}000{,}000}{0.000\ 000\ 000\ 000\ 0871} = \frac{3.251 \times 10^{11}}{8.71 \times 10^{-14}}$$

$= $ 3.732491389E24 — Calculator display

$= 3.73 \times 10^{24}$ — To three significant digits



› Figure 1

Figure 1 shows two solutions to this problem on a graphing calculator. In the first solution we entered the numbers in scientific notation, and in the second we used standard decimal notation. Although the multiple-line screen display on a graphing calculator allows us to enter very long standard decimals, scientific notation is usually more efficient and less prone to errors in data entry. Furthermore, as Figure 1 shows, the calculator uses scientific notation to display the answer, regardless of the manner in which the numbers are entered.

MATCHED PROBLEM 7

Repeat Example 7 for:

$$\frac{0.000\ 000\ 006\ 932}{62{,}600{,}000{,}000}$$

EXAMPLE 8

Measuring Time with an Atomic Clock

An atomic clock that counts the radioactive emissions of cesium is used to provide a precise definition of a second. One second is defined to be the time it takes cesium to emit 9,192,631,770 cycles of radiation. How many of these cycles will occur in 1 hour? Express the answer to five significant digits in scientific notation.

SOLUTION

$(9{,}192{,}631{,}770)(60^2) = $ 3.309347437E13

$= 3.3093 \times 10^{13}$

*For those not familiar with the meaning of *significant digits,* see Appendix A for a brief discussion of this concept.

MATCHED PROBLEM 8

Refer to Example 8. How many of these cycles will occur in 1 year? Express the answer to five significant digits in scientific notation.

EXAMPLE 9 Evaluating Rational Exponential Forms with a Calculator

Evaluate to four significant digits using a calculator. (Refer to the instruction book for your particular calculator to see how exponential forms are evaluated.)

(A) $11^{3/4}$ (B) $3.1046^{-2/3}$ (C) $(0.000\ 000\ 008\ 437)^{3/11}$

SOLUTIONS

(A) First change $\frac{3}{4}$ to the standard decimal form 0.75; then evaluate $11^{0.75}$ using a calculator.

$$11^{3/4} = 6.040$$

(B) $3.1046^{-2/3} = 0.4699$

(C) $(0.000\ 000\ 008\ 437)^{3/11} = (8.437 \times 10^{-9})^{3/11}$
$= 0.006\ 281$

MATCHED PROBLEM 9

Evaluate to four significant digits using a calculator.

(A) $2^{3/8}$ (B) $57.28^{-5/6}$ (C) $(83{,}240{,}000{,}000)^{5/3}$

ANSWERS TO MATCHED PROBLEMS

1. (A) 1 (B) 1 (C) 0.000 01 (D) 1,000 (E) x^4 (F) v^3/u^7
2. (A) $15x$ (B) $3/(2y^3)$ (C) $-14x^4$
3. (A) $y^6/(9x^8)$ (B) y^{12}/x^6 (C) $2n^5/(5m)$ (D) x^9 (E) $(a - b)^2$
4. (A) 2 (B) -2 (C) Not real (D) 2 (E) -2 (F) 0
5. (A) 27 (B) 81 (C) $10y^{13/12}$ (D) $16y/x^3$ (E) $2/x^{1/18}$
(F) $2x - 5x^{1/2}y^{1/2} - 3y$
6. (A) 4.3×10^2; 2.3×10^4; 3.45×10^8; 3×10^{-1}; 3.1×10^{-3}; 6.83×10^{-7}
(B) 4,000; 530,000; 0.0253; 0.000 007 42
7. 1.11×10^{-19} **8.** 2.8990×10^{17}
9. (A) 1.297 (B) 0.034 28 (C) 1.587×10^{18}

R-2 Exercises

All variables are restricted to prevent division by 0.

In Problems 1–26, evaluate each expression that results in a rational number.

1. 2^8 **2.** 3^8 **3.** $\left(\frac{2}{3}\right)^4$

4. $\left(\frac{3}{5}\right)^3$ **5.** 4^{-4} **6.** 2^{-6}

7. $(-5)^4$ **8.** $(-2)^5$ **9.** $(-3)^{-1}$

10. $(-3)^{-2}$ **11.** -7^{-2} **12.** -7^0

13. $\left(\frac{1}{3}\right)^0$ **14.** $\left(\frac{1}{5}\right)^{-1}$ **15.** $100^{1/2}$

16. $361^{1/2}$ **17.** $125^{1/3}$ **18.** $27^{2/3}$

19. $9^{-3/2}$ **20.** $2^{-1/2}$ **21.** $\left(\frac{1}{2}\right)^{-1/2}$

22. $64^{-4/3}$ **23.** $1^{-3} + 3^{-1}$ **24.** $4^{-1} + 4^2$

25. $\frac{5}{5^4}$ **26.** $\frac{8}{8^{-2}}$

Simplify Problems 27–38 and express answers using positive exponents only.

27. x^5x^{-2} **28.** y^6y^{-8} **29.** $(2y)(3y^2)(5y^4)$

30. $(6x^3)(4x^7)(x^{-5})$ **31.** $(a^2b^3)^5$ **32.** $(2c^4d^{-2})^{-3}$

33. $u^{1/3}u^{5/3}$ **34.** $v^{-1/5}v^{6/5}$ **35.** $(x^{-3})^{1/6}$

36. $(y^{-2/3})^{9/4}$ **37.** $(49a^4b^{-2})^{1/2}$ **38.** $(125a^{-9}b^{12})^{1/3}$

Write the numbers in Problems 39–44 in scientific notation.

39. 45,320,000 **40.** 3,670

41. 0.066 **42.** 0.029

43. 0.000 000 084 **44.** 0.000 497

In Problems 45–50, write each number in standard decimal form.

45. 9×10^{-5} **46.** 3×10^{-3}

47. 3.48×10^6 **48.** 8.63×10^8

49. 4.2×10^{-9} **50.** 1.6×10^{-7}

Simplify Problems 51–62, and write the answers using positive exponents only.

51. $\left(\frac{x^4y^{-1}}{x^{-2}y^3}\right)^2$ **52.** $\left(\frac{m^{-2}n^3}{m^4n^{-1}}\right)^2$ **53.** $\left(\frac{2x^{-3}y^2}{4xy^{-1}}\right)^{-2}$

54. $\left(\frac{6mn^{-2}}{3m^{-1}n^2}\right)^{-3}$ **55.** $\left(\frac{a^{-3}}{b^4}\right)^{1/12}$ **56.** $\left(\frac{m^{-2/3}}{n^{-1/2}}\right)^{-6}$

57. $\left(\frac{4x^{-2}}{y^4}\right)^{-1/2}$ **58.** $\left(\frac{w^4}{9x^{-2}}\right)^{-1/2}$

59. $\left(\frac{8a^{-4}b^3}{27a^2b^{-3}}\right)^{1/3}$ **60.** $\left(\frac{25x^5y^{-1}}{16x^{-3}y^{-5}}\right)^{1/2}$

*__61.__ $-3(x^3 + 3)^{-4}(3x^2)$ **62.** $-2(x^2 + 3x)^{-3}(2x + 3)$

63. What is the result of entering 2^{3^2} on a calculator?

64. Refer to Problem 63. What is the difference between $2^{(3^2)}$ and $(2^3)^2$? Which agrees with the value of 2^{3^2} obtained with a calculator?

65. If $n = 0$, then property 1 in Theorem 1 implies that $a^ma^0 = a^{m+0} = a^m$. Explain how this helps motivate the definition of a^0.

66. If $m = -n$, then property 1 in Theorem 1 implies that $a^{-n}a^n = a^0 = 1$. Explain how this helps motivate the definition of a^{-n}.

Evaluate Problems 67–70, to three significant digits using scientific notation and a calculator.

67. $\frac{(32.7)(0.000\ 000\ 008\ 42)}{(0.0513)(80{,}700{,}000{,}000)}$

68. $\frac{(4{,}320)(0.000\ 000\ 000\ 704)}{(835)(635{,}000{,}000{,}000)}$

69. $\frac{(5{,}760{,}000{,}000)}{(527)(0.000\ 007\ 09)}$

70. $\frac{0.000\ 000\ 007\ 23}{(0.0933)(43{,}700{,}000{,}000)}$

*The symbol ∫ denotes problems that are related to calculus.

In Problems 71–78, evaluate to four significant digits using a calculator. (Refer to the instruction book for your calculator to see how exponential forms are evaluated.)

71. $15^{5/4}$ **72.** $22^{3/2}$ **73.** $103^{-3/4}$

74. $827^{-3/8}$ **75.** $2.876^{8/5}$ **76.** $37.09^{7/3}$

77. $(0.000\ 000\ 077\ 35)^{-2/7}$ **78.** $(491{,}300{,}000{,}000)^{7/4}$

Problems 79–82, illustrate common errors involving rational exponents. In each case, find numerical examples that show that the left side is not always equal to the right side.

79. $(x + y)^{1/2} \neq x^{1/2} + y^{1/2}$ **80.** $(x^3 + y^3)^{1/3} \neq x + y$

81. $(x + y)^{1/3} \neq \dfrac{1}{(x + y)^3}$ **82.** $(x + y)^{-1/2} \neq \dfrac{1}{(x + y)^2}$

In Problems 83–86, m and n represent positive integers. Simplify and express answers using positive exponents.

83. $(a^{3/n}b^{3/m})^{1/3}$ **84.** $(a^{n/2}b^{n/3})^{1/n}$

85. $(x^{m/4}y^{n/3})^{-12}$ **86.** $(a^{m/3}b^{n/2})^{-6}$

87. If possible, find a real value of x such that:
(A) $(x^2)^{1/2} \neq x$ (B) $(x^2)^{1/2} = x$ (C) $(x^3)^{1/3} \neq x$

88. If possible, find a real value of x such that:
(A) $(x^2)^{1/2} \neq -x$ (B) $(x^2)^{1/2} = -x$ (C) $(x^3)^{1/3} = -x$

89. If n is even and b is negative, then $b^{1/n}$ is not real. If m is odd, n is even, and b is negative, is $(b^m)^{1/n}$ real?

90. If we assume that m is odd and n is even, is it possible that one of $(b^{1/n})^m$ and $(b^m)^{1/n}$ is real and the other is not?

APPLICATIONS

91. EARTH SCIENCE If the mass of Earth is approximately 6.1×10^{27} grams and each gram is 2.2×10^{-3} pound, what is the mass of Earth in pounds? Express the answer to two significant digits in scientific notation.

92. BIOLOGY In 1929 Vernadsky, a biologist, estimated that all the free oxygen of the earth weighs 1.5×10^{21} grams and that it is produced by life alone. If 1 gram is approximately 2.2×10^{-3} pound, what is the weight of the free oxygen in pounds? Express the answer to two significant digits in scientific notation.

93. COMPUTER SCIENCE If a computer can perform a single operation in 10^{-10} second, how many operations can it perform in 1 second? In 1 minute?

★**94.** COMPUTER SCIENCE If electricity travels in a computer circuit at the speed of light (1.86×10^5 miles per second), how far will electricity travel in the superconducting computer (see Problem 93) in the time it takes it to perform one operation? (Size of circuits is a critical problem in computer design.) Give the answer in miles, feet, and inches (1 mile = 5,280 feet). Compute answers to three significant digits.

95. ECONOMICS If in the United States in 2003 the national debt was about \$6,760,000,000,000 and the population was about 291,000,000, estimate to three significant digits each individual's share of the national debt. Write your answer in scientific notation and in standard decimal form.

96. ECONOMICS If in the United States in 2003 the gross domestic product (GDP) was about \$10,990,000,000,000 and the population was about 291,000,000, estimate to three significant digits the GDP per person. Write your answer in scientific notation and in standard decimal form.

97. ECONOMICS The number of units N of a finished product produced from the use of x units of labor and y units of capital for a particular Third World country is approximated by

$$N = 10x^{3/4}y^{1/4} \quad \text{Cobb-Douglas equation}$$

Estimate how many units of a finished product will be produced using 256 units of labor and 81 units of capital.

98. ECONOMICS The number of units N of a finished product produced by a particular automobile company where x units of labor and y units of capital are used is approximated by

$$N = 50x^{1/2}y^{1/2} \quad \text{Cobb-Douglas equation}$$

Estimate how many units will be produced using 256 units of labor and 144 units of capital.

99. BRAKING DISTANCE R. A. Moyer of Iowa State College found, in comprehensive tests carried out on 41 wet pavements, that the braking distance d (in feet) for a particular automobile traveling at v miles per hour was given approximately by

$$d = 0.0212v^{7/3}$$

Approximate the braking distance to the nearest foot for the car traveling on wet pavement at 70 miles per hour.

100. BRAKING DISTANCE Approximately how many feet would it take the car in Problem 99 to stop on wet pavement if it were traveling at 50 miles per hour? (Compute answer to the nearest foot.)

R-3 Radicals

- From Rational Exponents to Radicals, and Vice Versa
- Properties of Radicals
- Simplifying Radicals
- Rationalizing Denominators

What do the following algebraic expressions have in common?

$$2^{1/2} \qquad 2x^{2/3} \qquad \frac{1}{x^{1/2} + y^{1/2}}$$

$$\sqrt{2} \qquad 2\sqrt[3]{x^2} \qquad \frac{1}{\sqrt{x} + \sqrt{y}}$$

Each vertical pair represents the same quantity, one in rational exponent form and the other in *radical form*. There are occasions when it is more convenient to work with radicals than with rational exponents, or vice versa. In this section we see how the two forms are related and investigate some basic operations on radicals.

From Rational Exponents to Radicals, and Vice Versa

We start this discussion by defining an ***n*th-root radical:**

DEFINITION 1 $\sqrt[n]{b}$, *n*th-Root Radical

For n a natural number greater than 1 and b a real number, we define $\sqrt[n]{b}$ to be the **principal *n*th root of *b*** (see Definition 3 in Section R-2); that is,

$$\sqrt[n]{b} = b^{1/n}$$

The symbol $\sqrt{}$ is called a **radical,** n is called the **index,** and b is called the **radicand.**

If $n = 2$, we write $\sqrt{b}$ in place of $\sqrt[2]{b}$.

$\sqrt{25} = 25^{1/2} = 5$ $\qquad$ $\sqrt[5]{32} = 32^{1/5} = 2$

$-\sqrt{25} = -25^{1/2} = -5$ $\qquad$ $\sqrt[5]{-32} = (-32)^{1/5} = -2$

$\sqrt{-25}$ is not real $\qquad$ $\sqrt[4]{0} = 0^{1/4} = 0$

As already stated, it is often an advantage to be able to shift back and forth between rational exponent forms and radical forms. The following relationships, which are direct consequences of Definition 4 and Theorem 4 in Section R-2, are useful in this regard:

RATIONAL EXPONENT/RADICAL CONVERSIONS

For m and n positive integers ($n > 1$), and b not negative when n is even,

$$b^{m/n} = \begin{cases} (b^m)^{1/n} = \sqrt[n]{b^m} \\ (b^{1/n})^m = (\sqrt[n]{b})^m \end{cases}$$

$$2^{2/3} = \begin{cases} \sqrt[3]{2^2} \\ (\sqrt[3]{2})^2 \end{cases}$$

Note: Unless stated to the contrary, all variables in the rest of the discussion represent positive real numbers.

EXPLORE-DISCUSS 1

In each of the following, evaluate both radical forms.

$$16^{3/2} = \sqrt{16^3} = (\sqrt{16})^3$$
$$27^{2/3} = \sqrt[3]{27^2} = (\sqrt[3]{27})^2$$

Which radical conversion form is easier to use if you are performing the calculations by hand?

EXAMPLE 1

Rational Exponent/Radical Conversions

Change from rational exponent form to radical form.

(A) $x^{1/7} = \sqrt[7]{x}$

(B) $(3u^2v^3)^{3/5} = \sqrt[5]{(3u^2v^3)^3}$ or $(\sqrt[5]{3u^2v^3})^3$ The first is usually preferred.

(C) $y^{-2/3} = \dfrac{1}{y^{2/3}} = \dfrac{1}{\sqrt[3]{y^2}}$ or $\sqrt[3]{y^{-2}}$ or $\sqrt[3]{\dfrac{1}{y^2}}$

Change from radical form to rational exponent form.

(D) $\sqrt[5]{6} = 6^{1/5}$ (E) $-\sqrt[3]{x^2} = -x^{2/3}$ (F) $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

MATCHED PROBLEM 1

Change from rational exponent form to radical form.

(A) $u^{1/5}$ (B) $(6x^2y^5)^{2/9}$ (C) $(3xy)^{-3/5}$

Change from radical form to rational exponent form.

(D) $\sqrt[4]{9u}$ (E) $-\sqrt[7]{(2x)^4}$ (F) $\sqrt[3]{x^3 + y^3}$

Properties of Radicals

The exponent properties considered earlier imply the following properties of radicals.

THEOREM 1 Properties of Radicals

For n a natural number greater than 1, and x and y positive real numbers:

1. $\sqrt[n]{x^n} = x$ $\quad\sqrt[3]{x^3} = x$
2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\quad\sqrt[5]{xy} = \sqrt[5]{x}\sqrt[5]{y}$
3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$ $\quad\sqrt[4]{\dfrac{x}{y}} = \dfrac{\sqrt[4]{x}}{\sqrt[4]{y}}$

EXAMPLE 2

Simplifying Radicals

Simplify:

(A) $\sqrt[5]{(3x^2y)^5} = 3x^2y$

(B) $\sqrt{10}\sqrt{5} = \sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

(C) $\sqrt[3]{\dfrac{x}{27}} = \dfrac{\sqrt[3]{x}}{\sqrt[3]{27}} = \dfrac{\sqrt[3]{x}}{3}$ or $\dfrac{1}{3}\sqrt[3]{x}$

MATCHED PROBLEM 2

Simplify:

(A) $\sqrt[7]{(u^2 + v^2)^7}$ (B) $\sqrt{6}\sqrt{2}$ (C) $\sqrt[3]{\dfrac{x^2}{8}}$

››› **CAUTION** ›››

In general, properties of radicals can be used to simplify terms raised to powers, not sums of terms raised to powers. Thus, for x and y positive real numbers,

$$\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2} = x + y$$

but

$$\sqrt{x^2 + 2xy + y^2} = \sqrt{(x + y)^2} = x + y$$

› Simplifying Radicals

The properties of radicals provide us with the means of changing algebraic expressions containing radicals to a variety of equivalent forms. One form that is often useful is a *simplified form.* An algebraic expression that contains radicals is said to be in **simplified form** if all four of the conditions listed in the following definition are satisfied.

› DEFINITION 2 Simplified (Radical) Form

1. No radicand (the expression within the radical sign) contains a factor to a power greater than or equal to the index of the radical.
   For example, $\sqrt{x^5}$ violates this condition.
2. No power of the radicand and the index of the radical have a common factor other than 1.
   For example, $\sqrt[6]{x^4}$ violates this condition.
3. No radical appears in a denominator.
   For example, $y/\sqrt{x}$ violates this condition.
4. No fraction appears within a radical.
   For example, $\sqrt{\frac{3}{5}}$ violates this condition.

EXAMPLE 3 Finding Simplified Form

Express radicals in simplified form.

(A) $\sqrt{12x^3y^5z^2} = \sqrt{(4x^2y^4z^2)(3xy)}$ $\quad x^{pm}y^{pn} = (x^my^n)^p$

$= \sqrt{(2xy^2z)^2(3xy)}$ $\quad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$= \sqrt{(2xy^2z)^2}\sqrt{3xy}$ $\quad \sqrt[n]{x^n} = x$

$= 2xy^2z\sqrt{3xy}$

(B) $\sqrt[3]{6x^2y}\sqrt[3]{4x^5y^2} = \sqrt[3]{(6x^2y)(4x^5y^2)}$ Simplify radicand.

$= \sqrt[3]{24x^7y^3}$ Condition 1 is not met.

$= \sqrt[3]{(8x^6y^3)(3x)}$ $x^{pm}y^{pn} = (x^m y^n)^p$

$= \sqrt[3]{(2x^2y)^3(3x)}$ $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$= \sqrt[3]{(2x^2y)^3}\sqrt[3]{3x}$ $\sqrt[n]{x^n} = x$

$= 2x^2y\sqrt[3]{3x}$

(C) $\sqrt[6]{16x^4y^2} = [(4x^2y)^2]^{1/6}$ Condition 2 is not met.

$= (4x^2y)^{2/6}$ Note the convenience of using rational exponents.

$= (4x^2y)^{1/3}$ Write as a radical.

$= \sqrt[3]{4x^2y}$

(D) $\sqrt[3]{\sqrt{27}} = [(3^3)^{1/2}]^{1/3}$ $(x^m)^n = x^{mn}$

$= (3^3)^{1/6} = 3^{3/6}$ Write as a radical.

$= 3^{1/2} = \sqrt{3}$

MATCHED PROBLEM 3

Express radicals in simplified form.

(A) $\sqrt{18x^5y^2z^3}$ (B) $\sqrt[4]{27a^3b^3}\sqrt[4]{3a^5b^3}$ (C) $\sqrt[9]{8x^6y^3}$ (D) $\sqrt{\sqrt[3]{4}}$

Algebraic expressions involving radicals often can be simplified by adding and subtracting terms that contain exactly the same radical expressions. The distributive property of real numbers plays a central role in this process.

EXAMPLE 4

Combining Like Terms

Combine as many terms as possible:

(A) $5\sqrt{3} + 4\sqrt{3} = (5 + 4)\sqrt{3} = 9\sqrt{3}$

(B) $2\sqrt[3]{xy^2} - 7\sqrt[3]{xy^2} = (2 - 7)\sqrt[3]{xy^2} = -5\sqrt[3]{xy^2}$

(C) $3\sqrt{xy} - 2\sqrt[3]{xy} + 4\sqrt{xy} - 7\sqrt[3]{xy} = 3\sqrt{xy} + 4\sqrt{xy} - 2\sqrt[3]{xy} - 7\sqrt[3]{xy}$

$= 7\sqrt{xy} - 9\sqrt[3]{xy}$

MATCHED PROBLEM 4

Combine as many terms as possible:

(A) $6\sqrt{2} + 2\sqrt{2}$ (B) $3\sqrt[5]{2x^2y^3} - 8\sqrt[5]{2x^2y^3}$

(C) $5\sqrt[3]{mn^2} - 3\sqrt{mn} - 2\sqrt[3]{mn^2} + 7\sqrt{mn}$

EXAMPLE 5

Multiplication with Radical Forms

Multiply and simplify:

(A) $\sqrt{2}(\sqrt{10} - 3) = \sqrt{2}\sqrt{10} - \sqrt{2} \cdot 3 = \sqrt{20} - 3\sqrt{2} = 2\sqrt{5} - 3\sqrt{2}$

(B) $(\sqrt{2} - 3)(\sqrt{2} + 5) = \sqrt{2}\sqrt{2} - 3\sqrt{2} + 5\sqrt{2} - 15$
$= 2 + 2\sqrt{2} - 15$
$= 2\sqrt{2} - 13$

(C) $(\sqrt{x} - 3)(\sqrt{x} + 5) = \sqrt{x}\sqrt{x} - 3\sqrt{x} + 5\sqrt{x} - 15$
$= x + 2\sqrt{x} - 15$

(D) $(\sqrt[3]{m} + \sqrt[3]{n^2})(\sqrt[3]{m^2} - \sqrt[3]{n}) = \sqrt[3]{m^3} + \sqrt[3]{m^2n^2} - \sqrt[3]{mn} - \sqrt[3]{n^3}$
$= m - \sqrt[3]{mn} + \sqrt[3]{m^2n^2} - n$

MATCHED PROBLEM 5

Multiply and simplify:

(A) $\sqrt{3}(\sqrt{6} - 4)$ (B) $(\sqrt{3} - 2)(\sqrt{3} + 4)$

(C) $(\sqrt{y} - 2)(\sqrt{y} + 4)$ (D) $(\sqrt[3]{x^2} - \sqrt[3]{y^2})(\sqrt[3]{x} + \sqrt[3]{y})$

Rationalizing Denominators

Eliminating a radical from a denominator is referred to as **rationalizing the denominator.** To rationalize the denominator, we multiply the numerator and denominator by a suitable factor that will rationalize the denominator—that is, will leave the denominator free of radicals. This factor is called a **rationalizing factor.** The following special products are of use in finding some rationalizing factors (see Examples 6C, D):

$$(a - b)(a + b) = a^2 - b^2 \quad (1)$$
$$(a - b)(a^2 + ab + b^2) = a^3 - b^3 \quad (2)$$
$$(a + b)(a^2 - ab + b^2) = a^3 + b^3 \quad (3)$$

EXPLORE-DISCUSS 2

Use the preceding special products (1) to (3) to find a rationalizing factor for each of the following:

(A) $\sqrt{a} - \sqrt{b}$ **(B)** $\sqrt{a} + \sqrt{b}$ **(C)** $\sqrt[3]{a} - \sqrt[3]{b}$ **(D)** $\sqrt[3]{a} + \sqrt[3]{b}$

EXAMPLE 6 Rationalizing Denominators

Rationalize denominators.

(A) $\dfrac{3}{\sqrt{5}}$ (B) $\sqrt[3]{\dfrac{2a^2}{3b^2}}$ (C) $\dfrac{\sqrt{x} + \sqrt{y}}{3\sqrt{x} - 2\sqrt{y}}$ (D) $\dfrac{1}{\sqrt[3]{m} + 2}$

SOLUTIONS

(A) $\sqrt{5}$ is a rationalizing factor for $\sqrt{5}$, since $\sqrt{5}\sqrt{5} = \sqrt{5^2} = 5$. Thus, we multiply the numerator and denominator by $\sqrt{5}$ to rationalize the denominator:

$$\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

(B) $$\sqrt[3]{\frac{2a^2}{3b^2}} = \frac{\sqrt[3]{2a^2}}{\sqrt[3]{3b^2}} = \frac{\sqrt[3]{2a^2}\sqrt[3]{3^2b}}{\sqrt[3]{3b^2}\sqrt[3]{3^2b}} = \frac{\sqrt[3]{2 \cdot 3^2a^2b}}{\sqrt[3]{3^3b^3}} = \frac{\sqrt[3]{18a^2b}}{3b}$$

(C) Special product (1) suggests that if we multiply the denominator $3\sqrt{x} - 2\sqrt{y}$ by $3\sqrt{x} + 2\sqrt{y}$, we will obtain the difference of two squares and the denominator will be rationalized.

$$\frac{\sqrt{x} + \sqrt{y}}{3\sqrt{x} - 2\sqrt{y}} = \frac{(\sqrt{x} + \sqrt{y})(3\sqrt{x} + 2\sqrt{y})}{(3\sqrt{x} - 2\sqrt{y})(3\sqrt{x} + 2\sqrt{y})}$$ Expand.

$$= \frac{3\sqrt{x^2} + 2\sqrt{xy} + 3\sqrt{xy} + 2\sqrt{y^2}}{(3\sqrt{x})^2 - (2\sqrt{y})^2}$$ Combine like terms.

$$= \frac{3x + 5\sqrt{xy} + 2y}{9x - 4y}$$

(D) Special product (3) suggests that if we multiply the denominator $\sqrt[3]{m} + 2$ by $(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2$, we will obtain the sum of two cubes and the denominator will be rationalized.

$$\frac{1}{\sqrt[3]{m} + 2} = \frac{1[(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2]}{(\sqrt[3]{m} + 2)[(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2]}$$ Expand.

$$= \frac{\sqrt[3]{m^2} - 2\sqrt[3]{m} + 4}{(\sqrt[3]{m})^3 + 2^3}$$ $(\sqrt[n]{x})^n = x$

$$= \frac{\sqrt[3]{m^2} - 2\sqrt[3]{m} + 4}{m + 8}$$

MATCHED PROBLEM 6

Rationalize denominators.

(A) $\dfrac{6}{\sqrt{2x}}$ (B) $\dfrac{10x^3}{\sqrt[3]{4x}}$ (C) $\dfrac{\sqrt{x}+2}{2\sqrt{x}+3}$ (D) $\dfrac{1}{1-\sqrt[3]{y}}$

ANSWERS TO MATCHED PROBLEMS

1. (A) $\sqrt[5]{u}$ (B) $\sqrt[9]{(6x^2y^5)^2}$ or $(\sqrt[9]{6x^2y^5})^2$ (C) $1/\sqrt[5]{(3xy)^3}$ (D) $(9u)^{1/4}$
(E) $-(2x)^{4/7}$ (F) $(x^3+y^3)^{1/3}$
2. (A) u^2+v^2 (B) $2\sqrt{3}$ (C) $(\sqrt[3]{x^2})/2$ or $\frac{1}{2}\sqrt[3]{x^2}$
3. (A) $3x^2yz\sqrt{2xz}$ (B) $3a^2b\sqrt[4]{b^2}=3a^2b\sqrt{b}$ (C) $\sqrt[3]{2x^2y}$ (D) $\sqrt[3]{2}$
4. (A) $8\sqrt{2}$ (B) $-5\sqrt[5]{2x^2y^3}$ (C) $3\sqrt[3]{mn^2}+4\sqrt{mn}$
5. (A) $3\sqrt{2}-4\sqrt{3}$ (B) $2\sqrt{3}-5$ (C) $y+2\sqrt{y}-8$
(D) $x+\sqrt[3]{x^2y}-\sqrt[3]{xy^2}-y$
6. (A) $\dfrac{3\sqrt{2x}}{x}$ (B) $5x^2\sqrt[3]{2x^2}$ (C) $\dfrac{2x+\sqrt{x}-6}{4x-9}$ (D) $\dfrac{1+\sqrt[3]{y}+\sqrt[3]{y^2}}{1-y}$

R-3 Exercises

Unless stated to the contrary, all variables represent positive real numbers

In Problems 1–8, change to radical form. Do not simplify.

1. $32^{1/5}$
2. $625^{3/4}$
3. $8x^{2/3}$
4. $(3x^2y)^{3/7}$
5. $4x^{-1/2}$
6. $32y^{-2/5}$
7. $x^{1/3}-y^{1/3}$
8. $(x-y)^{1/3}$

In Problems 9–14, change to rational exponent form. Do not simplify.

9. $\sqrt{361}$
10. $\sqrt[3]{17^2}$
11. $4x\sqrt[5]{y^3}$
12. $\sqrt[4]{7x^3y^2}$
13. $\sqrt[3]{x^2+y^2}$
14. $\sqrt[3]{x^2}+\sqrt[3]{y^2}$

In Problems 15–34, write in simplified form.

15. $\sqrt[3]{-8}$
16. $\sqrt[3]{-27}$
17. $-\sqrt{128}$
18. $-\sqrt{125}$
19. $\sqrt{27}-5\sqrt{3}$
20. $2\sqrt{8}+\sqrt{18}$
21. $\sqrt[3]{5}-\sqrt[3]{25}+\sqrt[3]{625}$
22. $\sqrt{20}+\sqrt[3]{40}-\sqrt[3]{5}$
23. $\sqrt[3]{25}\sqrt[3]{10}$
24. $\sqrt{6}\sqrt{14}$
25. $\sqrt{9x^8y^4}$
26. $\sqrt{16m^4y^8}$
27. $\sqrt[4]{16m^4n^8}$
28. $\sqrt[5]{32a^{15}b^{10}}$
29. $\sqrt[4]{m^2}$
30. $\sqrt[10]{n^6}$
31. $\sqrt[5]{\sqrt[3]{xy}}$
32. $\sqrt{\sqrt[4]{5x}}$
33. $\sqrt[3]{9x^2}\sqrt[3]{9x}$
34. $\sqrt{2x}\sqrt{8xy}$

In Problems 35–46, rationalize denominators, and write in simplified form.

35. $\dfrac{1}{2\sqrt{5}}$
36. $\dfrac{1}{3\sqrt{7}}$
37. $\dfrac{1}{\sqrt[3]{7}}$
38. $\dfrac{10}{\sqrt[4]{5}}$
39. $\dfrac{2}{\sqrt{44}}$
40. $\dfrac{3}{\sqrt[3]{54}}$
41. $\dfrac{6x}{\sqrt{3x}}$
42. $\dfrac{12y^2}{\sqrt{6y}}$
43. $\dfrac{2}{\sqrt{2}-1}$
44. $\dfrac{4}{\sqrt{6}-2}$
45. $\dfrac{\sqrt{2}}{\sqrt{6}+2}$
46. $\dfrac{\sqrt{2}}{\sqrt{10}-2}$

In Problems 47–54, write in simplified form.

47. $x\sqrt[5]{3^6x^7y^{11}}$
48. $2a\sqrt[3]{8a^8b^{13}}$
49. $\dfrac{\sqrt[4]{32m^7n^9}}{2mn}$
50. $\dfrac{\sqrt[5]{32u^{12}v^8}}{uv}$
51. $\sqrt[3]{\sqrt[4]{a^9b^3}}$
52. $\sqrt{\sqrt[6]{x^8y^6}}$

53. $\sqrt[3]{2x^2y^4}\sqrt[3]{3x^5y}$ **54.** $\sqrt[4]{4m^5n}\sqrt[4]{6m^3n^4}$

In Problems 55–60, rationalize denominators and write in simplified form.

55. $\dfrac{\sqrt{2m}\sqrt{5}}{\sqrt{20m}}$ **56.** $\dfrac{\sqrt{6}\sqrt{8c}}{\sqrt{18c}}$ **57.** $\dfrac{3\sqrt{y}}{2\sqrt{y}-3}$

58. $\dfrac{5\sqrt{x}}{3-2\sqrt{x}}$ **59.** $\dfrac{2\sqrt{5}+3\sqrt{2}}{5\sqrt{5}+2\sqrt{2}}$ **60.** $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$

Problems 61–64 are calculus-related. Rationalize the numerators; that is, perform operations on the fractions that eliminate radicals from the numerators. (This is a particularly useful operation in some problems in calculus.)

61. $\dfrac{\sqrt{t}-\sqrt{x}}{t-x}$ **62.** $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

63. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ **64.** $\dfrac{\sqrt{2+h}+\sqrt{2}}{h}$

In Problems 65–72, evaluate to four significant digits using a calculator. (Read the instruction booklet accompanying your calculator for the process required to evaluate $\sqrt[n]{x}$.*)*

65. $\sqrt{0.032\ 965}$ **66.** $\sqrt{419.763}$

67. $\sqrt[3]{45.0218}$ **68.** $\sqrt[4]{0.098\ 553}$

69. $\sqrt[8]{5.477 \times 10^{-9}}$ **70.** $\sqrt[7]{4.892 \times 10^{16}}$

71. $\sqrt[5]{9+\sqrt[5]{9}}$ **72.** $\sqrt[3]{2+\sqrt[3]{2}}$

For what real numbers are Problems 73–76 true?

73. $\sqrt{x^2} = -x$ **74.** $\sqrt{x^2} = x$

75. $\sqrt[3]{x^3} = x$ **76.** $\sqrt[3]{x^3} = -x$

In Problems 77 and 78, evaluate each expression on a calculator and determine which pairs have the same value. Verify these results algebraically.

77. (A) $\sqrt{3}+\sqrt{5}$ (B) $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$
(C) $1+\sqrt{3}$ (D) $\sqrt[3]{10+6\sqrt{3}}$
(E) $\sqrt{8+\sqrt{60}}$ (F) $\sqrt{6}$

78. (A) $2\sqrt[3]{2+\sqrt{5}}$ (B) $\sqrt{8}$
(C) $\sqrt{3}+\sqrt{7}$ (D) $\sqrt{3+\sqrt{8}}+\sqrt{3-\sqrt{8}}$
(E) $\sqrt{10+\sqrt{84}}$ (F) $1+\sqrt{5}$

In Problems 79–82, rationalize denominators.

79. $\dfrac{1}{\sqrt[3]{a}-\sqrt[3]{b}}$ **80.** $\dfrac{1}{\sqrt[3]{m}+\sqrt[3]{n}}$

81. $\dfrac{1}{\sqrt{x}-\sqrt{y}+\sqrt{z}}$ **82.** $\dfrac{1}{\sqrt{x}+\sqrt{y}-\sqrt{z}}$

[*Hint for Problem 81:* Start by multiplying numerator and denominator by $(\sqrt{x}-\sqrt{y})-\sqrt{z}$.]

Problems 83 and 84 are calculus-related. Rationalize numerators.

83. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$ **84.** $\dfrac{\sqrt[3]{t}-\sqrt[3]{x}}{t-x}$

85. Show that $\sqrt[kn]{x^{km}} = \sqrt[n]{x^m}$ for k, m, and n natural numbers greater than 1.

86. Show that $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$ for m and n natural numbers greater than 1.

APPLICATIONS

87. PHYSICS—RELATIVISTIC MASS The mass M of an object moving at a velocity v is given by

$$M = \frac{M_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

where M_0 = mass at rest and c = velocity of light. The mass of an object increases with velocity and tends to infinity as the velocity approaches the speed of light. Show that M can be written in the form

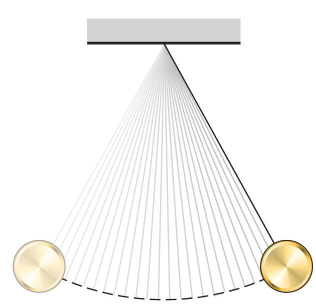
$$M = \frac{M_0 c\sqrt{c^2-v^2}}{c^2-v^2}$$

88. PHYSICS—PENDULUM A simple pendulum is formed by hanging a bob of mass M on a string of length L from a fixed support (see the figure). The time it takes the bob to swing from right to left and back again is called the **period** T and is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the gravitational constant. Show that T can be written in the form

$$T = \frac{2\pi\sqrt{gL}}{g}$$



R-4 Polynomials: Basic Operations

- Polynomials
- Combining Like Terms
- Addition and Subtraction
- Multiplication
- Application

In this section we review the basic operations on *polynomials*. Polynomials are expressions such as $x^4 - 5x^2 + 1$ or $3xy - 2x + 5y + 6$ that are built from constants and variables using only addition, subtraction, and multiplication (the power x^4 is the product $x \cdot x \cdot x \cdot x$). Polynomials are used throughout mathematics to describe and approximate mathematical relationships.

Polynomials

Algebraic expressions are formed by using constants and variables and the algebraic operations of addition, subtraction, multiplication, division, raising to powers, and taking roots. Some examples are

$$\begin{array}{ll} \sqrt[3]{x^3 + 5} & 5x^4 + 2x^2 - 7 \\ x + y - 7 & (2x - y)^2 \\ \dfrac{x - 5}{x^2 + 2x - 5} & 1 + \dfrac{1}{1 + \dfrac{1}{x}} \end{array}$$

An algebraic expression involving only the operations of addition, subtraction, multiplication, and raising to natural number powers is called a **polynomial.** (Note that raising to a natural number power is repeated multiplication.) Some examples are

$$\begin{array}{ll} 2x - 3 & 4x^2 - 3x + 7 \\ x - 2y & 5x^3 - 2x^2 - 7x + 9 \\ 5 & x^2 - 3xy + 4y^2 \\ 0 & x^3 - 3x^2y + xy^2 + 2y^7 \end{array}$$

In a polynomial, a variable cannot appear in a denominator, as an exponent, or within a radical. Accordingly, a **polynomial in one variable** x is constructed by adding or subtracting constants and terms of the form ax^n, where a is a real number and n is a natural number. A **polynomial in two variables** x and y is constructed by adding and subtracting constants and terms of the form ax^my^n, where a is a real number and m and n are natural numbers. Polynomials in three or more variables are defined in a similar manner.

Polynomials can be classified according to their *degree.* If a term in a polynomial has only one variable as a factor, then the **degree of that term** is the power of the variable. If two or more variables are present in a term as factors, then the **degree of the term** is the sum of the powers of the variables. The **degree of a polynomial** is the degree of the nonzero term with the highest degree in the polynomial. Any nonzero constant is defined to be a **polynomial of degree 0.** The number 0 is also a polynomial but is not assigned a degree.

EXAMPLE 1 Polynomials and Nonpolynomials

(A) Polynomials in one variable:

$$x^2 - 3x + 2 \qquad 6x^3 - \sqrt{2}x - \tfrac{1}{3}$$

(B) Polynomials in several variables:

$$3x^2 - 2xy + y^2 \qquad 4x^3y^2 - \sqrt{3}xy^2z^5$$

(C) Nonpolynomials:

$$\sqrt{2x} - \frac{3}{x} + 5 \qquad \frac{x^2 - 3x + 2}{x - 3} \qquad \sqrt{x^2 - 3x + 1}$$

(D) The degree of the first term in $6x^3 - \sqrt{2}x - \frac{1}{3}$ is 3, the degree of the second term is 1, the degree of the third term is 0, and the degree of the whole polynomial is 3.

(E) The degree of the first term in $4x^3y^2 - \sqrt{3}xy^2$ is 5, the degree of the second term is 3, and the degree of the whole polynomial is 5.

MATCHED PROBLEM 1

(A) Which of the following are polynomials?

$$3x^2 - 2x + 1 \qquad \sqrt{x - 3} \qquad x^2 - 2xy + y^2 \qquad \frac{x - 1}{x^2 + 2}$$

(B) Given the polynomial $3x^5 - 6x^3 + 5$, what is the degree of the first term? The second term? The whole polynomial?

(C) Given the polynomial $6x^4y^2 - 3xy^3$, what is the degree of the first term? The second term? The whole polynomial?

In addition to classifying polynomials by degree, we also call a single-term polynomial a **monomial,** a two-term polynomial a **binomial,** and a three-term polynomial a **trinomial.**

$\frac{5}{2}x^2y^3$ Monomial

$x^3 + 4.7$ Binomial

$x^4 - \sqrt{2}x^2 + 9$ Trinomial

› Combining Like Terms

We start with a word about *coefficients.* A constant in a term of a polynomial, including the sign that precedes it, is called the **numerical coefficient,** or simply, the **coefficient,** of the term. If a constant doesn't appear, or only a + sign appears, the coefficient is understood to be 1. If only a − sign appears, the coefficient is understood to be −1. Thus, given the polynomial

$$2x^4 - 4x^3 + x^2 - x + 5 \qquad 2x^4 + (-4)x^3 + 1x^2 + (-1)x + 5$$

the coefficient of the first term is 2, the coefficient of the second term is −4, the coefficient of the third term is 1, the coefficient of the fourth term is −1, and the coefficient of the last term is 5.

At this point, it is useful to state two additional distributive properties of real numbers that follow from the distributive properties stated in Section R-1.

› ADDITIONAL DISTRIBUTIVE PROPERTIES

1. $a(b - c) = (b - c)a = ab - ac$

2. $a(b + c + \cdots + f) = ab + ac + \cdots + af$

Two terms in a polynomial are called **like terms** if they have exactly the same variable factors to the same powers. The numerical coefficients may or may not be the same. Since constant terms involve no variables, all constant terms are like terms. If a polynomial contains two or more like terms, these terms can be combined into a single term by making use of distributive properties. Consider the following example:

$$\begin{aligned} 5x^3y - 2xy - x^3y - 2x^3y &= 5x^3y - x^3y - 2x^3y - 2xy && \text{Group like terms.} \\ &= (5x^3y - x^3y - 2x^3y) - 2xy && \text{Factor out } x^3y. \\ &= (5 - 1 - 2)x^3y - 2xy && \text{Simplify.} \\ &= 2x^3y - 2xy \end{aligned}$$

It should be clear that free use has been made of the real number properties discussed earlier. The steps done in the dashed box are usually done mentally, and the process is quickly mechanized as follows:

Like terms in a polynomial are combined by adding their numerical coefficients.

EXAMPLE 2 Simplifying Polynomials

Remove parentheses and combine like terms:

(A) $2(3x^2 - 2x + 5) + (x^2 + 3x - 7)$

$= 2(3x^2 - 2x + 5) + 1(x^2 + 3x - 7)$ Distribute, remove parentheses.

$= 6x^2 - 4x + 10 + x^2 + 3x - 7$ Combine like terms.

$= 7x^2 - x + 3$

(B) $(x^3 - 2x - 6) - (2x^3 - x^2 + 2x - 3)$

$= 1(x^3 - 2x - 6) + (-1)(2x^3 - x^2 + 2x - 3)$ Distribute, remove parentheses.

$= x^3 - 2x - 6 - 2x^3 + x^2 - 2x + 3$ Combine like terms.

$= -x^3 + x^2 - 4x - 3$

(C) $[3x^2 - (2x + 1)] - (x^2 - 1) = [3x^2 - 2x - 1] - (x^2 - 1)$ Remove parentheses.

$= 3x^2 - 2x - 1 - x^2 + 1$ Combine like terms.

$= 2x^2 - 2x$

MATCHED PROBLEM 2

Remove parentheses and combine like terms:

(A) $3(u^2 - 2v^2) + (u^2 + 5v^2)$

(B) $(m^3 - 3m^2 + m - 1) - (2m^3 - m + 3)$

(C) $(x^3 - 2) - [2x^3 - (3x + 4)]$

Addition and Subtraction

Addition and subtraction of polynomials can be thought of in terms of removing parentheses and combining like terms, as illustrated in Example 2. Horizontal and vertical arrangements are illustrated in the next two examples. You should be able to work either way, letting the situation dictate the choice.

EXAMPLE 3 Adding Polynomials

Add: $x^4 - 3x^3 + x^2$, $\quad -x^3 - 2x^2 + 3x$, $\quad$ and $\quad 3x^2 - 4x - 5$

SOLUTION

Add horizontally:

$(x^4 - 3x^3 + x^2) + (-x^3 - 2x^2 + 3x) + (3x^2 - 4x - 5)$ Remove parentheses.

$= x^4 - 3x^3 + x^2 - x^3 - 2x^2 + 3x + 3x^2 - 4x - 5$ Combine like terms.

$= x^4 - 4x^3 + 2x^2 - x - 5$

Or vertically, by lining up like terms and adding their coefficients:

$$\begin{array}{r} x^4 - 3x^3 + x^2 \\ - x^3 - 2x^2 + 3x \\ 3x^2 - 4x - 5 \\ \hline x^4 - 4x^3 + 2x^2 - x - 5 \end{array}$$

MATCHED PROBLEM 3

Add horizontally and vertically:

$$3x^4 - 2x^3 - 4x^2, \quad x^3 - 2x^2 - 5x, \quad \text{and} \quad x^2 + 7x - 2$$

EXAMPLE 4 **Subtracting Polynomials**

Subtract: $4x^2 - 3x + 5$ from $x^2 - 8$

SOLUTION

$$\begin{aligned} &(x^2 - 8) - (4x^2 - 3x + 5) \\ &= x^2 - 8 - 4x^2 + 3x - 5 \\ &= -3x^2 + 3x - 13 \end{aligned}$$

or

$$\begin{array}{r} x^2 - 8 \\ -4x^2 + 3x - 5 \\ \hline -3x^2 + 3x - 13 \end{array}$$

← Change signs and add.

MATCHED PROBLEM 4

Subtract: $2x^2 - 5x + 4$ from $5x^2 - 6$

››› CAUTION ›››

When you use a horizontal arrangement to subtract a polynomial with more than one term, you must enclose the polynomial in parentheses. Thus, to subtract $2x + 5$ from $4x - 11$, you must write

$$4x - 11 - (2x + 5) \quad \text{and not} \quad 4x - 11 - 2x + 5$$

› Multiplication

Multiplication of algebraic expressions involves the extensive use of distributive properties for real numbers, as well as other real number properties.

EXAMPLE 5

Multiplying Polynomials

Multiply: $(2x - 3)(3x^2 - 2x + 3)$

SOLUTION

$$
\begin{aligned}
&(2x - 3)(3x^2 - 2x + 3) \\
&\quad = 2x(3x^2 - 2x + 3) - 3(3x^2 - 2x + 3) && \text{Distribute, remove parentheses.} \\
&\quad = 6x^3 - 4x^2 + 6x - 9x^2 + 6x - 9 && \text{Combine like terms.} \\
&\quad = 6x^3 - 13x^2 + 12x - 9
\end{aligned}
$$

Or, using a vertical arrangement,

$$
\begin{array}{rrrr}
3x^2 & - 2x & + 3 & \\
2x & - 3 & & \\
\hline
6x^3 - & 4x^2 + & 6x & \\
- & 9x^2 + & 6x & - 9 \\
\hline
6x^3 - & 13x^2 + & 12x & - 9
\end{array}
$$

MATCHED PROBLEM 5

Multiply:

$$(2x - 3)(2x^2 + 3x - 2)$$

Thus, to multiply two polynomials, multiply each term of one by each term of the other, and combine like terms.

EXAMPLE 6

Multiplying Binomials

Multiply:

(A) $(2x - 3y)(5x + 2y) = 10x^2 + 4xy - 15xy - 6y^2 = 10x^2 - 11xy - 6y^2$

(B) $(3a - 2b)(3a + 2b) = (3a)^2 - (2b)^2 = 9a^2 - 4b^2$

(C) $(5x - 3)^2 = (5x)^2 - 2(5x)(3) + 3^2 = 25x^2 - 30x + 9$

(D) $(m + 2n)^2 = m^2 + 4mn + 4n^2$

MATCHED PROBLEM 6

Multiply:

(A) $(4u - 3v)(2u + v)$ (B) $(2xy + 3)(2xy - 3)$

(C) $(m + 4n)(m - 4n)$ (D) $(2u - 3v)^2$ (E) $(6x + y)^2$

Products of certain binomial factors occur so frequently that it is useful to remember formulas for their products. The following formulas are easily verified by multiplying the factors on the left:

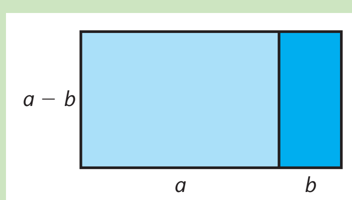
SPECIAL PRODUCTS

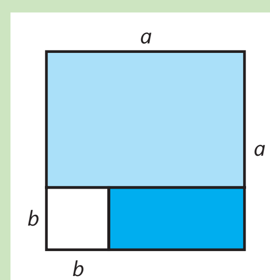
1. $(a - b)(a + b) = a^2 - b^2$
2. $(a + b)^2 = a^2 + 2ab + b^2$
3. $(a - b)^2 = a^2 - 2ab + b^2$

EXPLORE-DISCUSS 1

(A) Explain the relationship between special product formula 1 and the areas of the rectangles in the figures.

$$(a - b)(a + b) \qquad = \qquad a^2 - b^2$$





(B) Construct similar figures to provide geometric interpretations for special product formulas 2 and 3.

The following conventions govern the order in which algebraic operations are performed.

ORDER OF OPERATIONS

1. Simplify inside the innermost grouping first, then the next innermost, and so on.

$$2[3 - (x - 4)] = 2[3 - x + 4]$$
$$= 2(7 - x) = 14 - 2x$$

2. Unless grouping symbols indicate otherwise, apply exponents before multiplication or division is performed.

$$2(x - 2)^2 = 2(x^2 - 4x + 4) = 2x^2 - 8x + 8$$

3. Unless grouping symbols indicate otherwise, perform multiplication and division before addition and subtraction. In either case, proceed from left to right.

$$5 - 2(x - 3) = 5 - 2x + 6 = 11 - 2x$$

EXAMPLE 7

Combined Operations

Perform the indicated operations and simplify:

(A) $3x - \{5 - 3[x - x(3 - x)]\}$

$= 3x - \{5 - 3[x - 3x + x^2]\}$ Combine like terms.

$= 3x - \{5 - 3[-2x + x^2]\}$ Distribute, remove square brackets.

$= 3x - \{5 + 6x - 3x^2\}$ Distribute, remove braces.

$= 3x - 5 - 6x + 3x^2$ Combine like terms.

$= 3x^2 - 3x - 5$

(B) $(x - 2y)(2x + 3y) - (2x + y)^2$

$= 2x^2 + 3xy - 4xy - 6y^2 - (4x^2 + 4xy + y^2)$ Distribute, remove parentheses.

$= 2x^2 - xy - 6y^2 - 4x^2 - 4xy - y^2$ Combine like terms.

$= -2x^2 - 5xy - 7y^2$

(C) $(2m + 3n)^3 = (2m + 3n)(2m + 3n)^2$ Expand the square.

$= (2m + 3n)(4m^2 + 12mn + 9n^2)$ Expand.

$= 8m^3 + 24m^2n + 18mn^2 + 12m^2n$ Combine like terms.
$+ 36mn^2 + 27n^3$

$= 8m^3 + 36m^2n + 54mn^2 + 27n^3$

MATCHED PROBLEM 7

Perform the indicated operations and simplify:

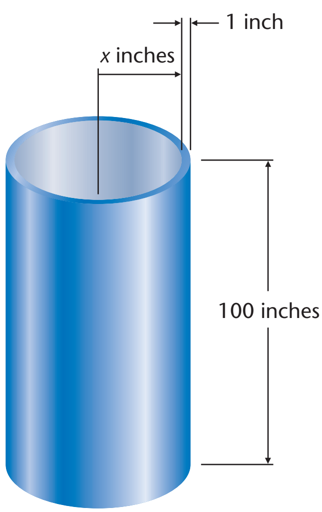
(A) $2t - \{7 - 2[t - t(4 + t)]\}$ (B) $(u - 3v)^2 - (2u - v)(2u + v)$

(C) $(4x - y)^3$

Application

EXAMPLE 8 Volume of a Cylindrical Shell

A plastic water pipe with a hollow center is 100 inches long, 1 inch thick, and has an inner radius of x inches (see the figure). Write an algebraic expression in terms of x that represents the volume of the plastic used to construct the pipe. Simplify the expression. [*Recall:* The volume V of a right circular cylinder of radius r and height h is given by $V = \pi r^2 h$.]



SOLUTION

A right circular cylinder with a hollow center is called a *cylindrical shell.* The volume of the shell is equal to the volume of the cylinder minus the volume of the hole. Since the radius of the hole is x inches and the pipe is 1 inch thick, the radius of the cylinder is $x + 1$ inches. Thus, we have

$$\left(\text{Volume of shell}\right) = \left(\text{Volume of cylinder}\right) - \left(\text{Volume of hole}\right)$$

$$\begin{aligned} \text{Volume} &= \pi(x + 1)^2 100 - \pi x^2 100 && \text{Expand the square.} \\ &= 100\pi(x^2 + 2x + 1) - 100\pi x^2 && \text{Distribute, remove parentheses.} \\ &= 100\pi x^2 + 200\pi x + 100\pi - 100\pi x^2 && \text{Combine like terms.} \\ &= 200\pi x + 100\pi \end{aligned}$$

MATCHED PROBLEM 8

A plastic water pipe is 200 inches long, 2 inches thick, and has an outer radius of x inches. Write an algebraic expression in terms of x that represents the volume of the plastic used to construct the pipe. Simplify the expression.

ANSWERS TO MATCHED PROBLEMS

1. (A) $3x^2 - 2x + 1, x^2 - 2xy + y^2$ (B) 5, 3, 5 (C) 6, 4, 6
2. (A) $4u^2 - v^2$ (B) $-m^3 - 3m^2 + 2m - 4$ (C) $-x^3 + 3x + 2$
3. $3x^4 - x^3 - 5x^2 + 2x - 2$ **4.** $3x^2 + 5x - 10$ **5.** $4x^3 - 13x + 6$
6. (A) $8u^2 - 2uv - 3v^2$ (B) $4x^2y^2 - 9$ (C) $m^2 - 16n^2$
(D) $4u^2 - 12uv + 9v^2$ (E) $36x^2 + 12xy + y^2$
7. (A) $-2t^2 - 4t - 7$ (B) $-3u^2 - 6uv + 10v^2$ (C) $64x^3 - 48x^2y + 12xy^2 - y^3$
8. Volume $= 200\pi x^2 - 200\pi(x - 2)^2 = 800\pi x - 800\pi$

R-4 Exercises

Problems 1–8 refer to the following polynomials:
(a) $x^4 - 2x^2 + 3$ and (b) $x^3 - 1$.

1. What is the degree of (a)?

2. What is the degree of (b)?

3. What is the degree of the sum of (a) and (b)?

4. What is the degree of the product of (a) and (b)?

5. Multiply (a) and (b).

6. Add (a) and (b).

7. Subtract (b) from (a).

8. Subtract (a) from (b).

In Problems 9–14, is the algebraic expression a polynomial? If so, give its degree.

9. $6 - 5x + x^2$

10. $3x^2 + 5x^4 - 8$

11. $x^3 - 5x + 4\sqrt{x}$

12. $x^5 + 7x^3 + \sqrt{2}$

13. $x^3 + 9x^2 + 5^{-2}$

14. $2x^4 + 3x^{-1} + 10$

In Problems 15–36, perform the indicated operations and simplify.

15. $2(x - 1) + 3(2x - 3) - (4x - 5)$

16. $2(u - 1) - (3u + 2) - 2(2u - 3)$

17. $2y - 3y[4 - 2(y - 1)]$

18. $4a - 2a[5 - 3(a + 2)]$

19. $(m - n)(m + n)$

20. $(a + b)(a - b)$

21. $(4t - 3)(t - 2)$

22. $(3x - 5)(2x + 1)$

23. $(5y - 1)(3 - 2y)$

24. $(-2 + m)(6m + 5)$

25. $(3x + 2y)(x - 3y)$

26. $(2x - 3y)(x + 2y)$

27. $(2m - 7)(2m + 7)$

28. $(3y + 2)(3y - 2)$

29. $(6x - 4y)(5x + 3y)$

30. $(3m + 7n)(2m - 5n)$

31. $(3x - 2y)(3x + 2y)$

32. $(4m + 3n)(4m - 3n)$

33. $(4x - y)^2$

34. $(3u + 4v)^2$

35. $(a + b)(a^2 - ab + b^2)$

36. $(a - b)(a^2 + ab + b^2)$

In Problems 37–50, perform the indicated operations and simplify.

37. $2x - 3\{x + 2[x - (x + 5)] + 1\}$

38. $m - \{m - [m - (m - 1)]\}$

39. $2\{3[a - 4(1 - a)] - (5 - a)\}$

40. $5b - 3\{-[2 - 4(2b - 1)] + 2(2 - 3b)\}$

41. $(2x^2 - 3x + 1)(x^2 + x - 2)$

42. $(x^2 - 3xy + y^2)(x^2 + 3xy + y^2)$

43. $(x - 2y)^2(x + 2y)^2$

44. $(n^2 + 4nm + m^2)(n^2 - 4nm + m^2)$

45. $(3u - 2v)^2 - (2u - 3v)(2u + 3v)$

46. $(2a - b)^2 - (a + 2b)^2$

47. $(z + 2)(z^2 - 2z + 3) + z - 7$

48. $(y + 3)(y^2 - 3y + 1) + 8y - 1$

49. $(2m - n)^3$

50. $(3a + 2b)^3$

Problems 51–58 are calculus-related. Perform the indicated operations and simplify.

51. $3(x + h) - 7 - (3x - 7)$

52. $(x + h)^2 - x^2$

53. $2(x + h)^2 - 3(x + h) - (2x^2 - 3x)$

54. $-4(x + h)^2 + 6(x + h) - (-4x^2 + 6x)$

55. $2(x + h)^2 - 4(x + h) - 9 - (2x^2 - 4x - 9)$

56. $3(x + h)^2 + 5(x + h) + 7 - (3x^2 + 5x + 7)$

57. $(x + h)^3 - 2(x + h)^2 - (x^3 - 2x^2)$

58. $(x + h)^3 + 3(x + h) - (x^3 + 3x)$

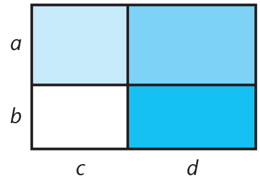
59. Subtract the sum of the first two polynomials from the sum of the last two: $3m^2 - 2m + 5$, $4m^2 - m$, $3m^2 - 3m - 2$, $m^3 + m^2 + 2$.

60. Subtract the sum of the last two polynomials from the sum of the first two: $2x^2 - 4xy + y^2$, $3xy - y^2$, $x^2 - 2xy - y^2$, $-x^2 + 3xy - 2y^2$.

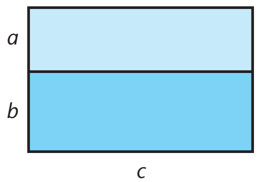
61. Explain the relationship between the equation

$$(a + b)(c + d) = ac + ad + bc + bd$$

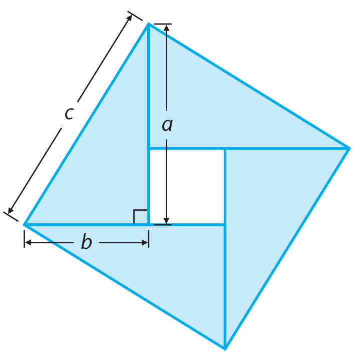
and the rectangles in the figure.



62. Explain the relationship between the distributive property of real numbers and the areas of the rectangles in the figure.



63. In the figure, the area of the large square is equal to the sum of the area of the small square and the areas of the four congruent triangles. Translate this verbal statement into a mathematical equation and simplify. What famous theorem have you proven?



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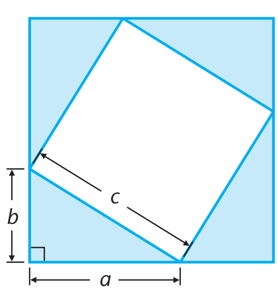


In Problems 65–68, perform the indicated operations and simplify.

65. $2(x - 2)^3 - (x - 2)^2 - 3(x - 2) - 4$

66. $(2x - 1)^3 - 2(2x - 1)^2 + 3(2x - 1) + 7$

67. $-3x\{x[x - x(2 - x)] - (x + 2)(x^2 - 3)\}$

68. $2\{(x - 3)(x^2 - 2x + 1) - x[3 - x(x - 2)]\}$

69. Show by example that, in general, $(a + b)^2 \neq a^2 + b^2$. Discuss possible conditions on a and b that would make this a valid equation.

70. Show by example that, in general, $(a - b)^2 \neq a^2 - b^2$. Discuss possible conditions on a and b that would make this a valid equation.

71. If you are given two polynomials, one of degree m and the other of degree n, $m > n$, what is the degree of the sum?

72. What is the degree of the product of the two polynomials in Problem 71?

73. How does the answer to Problem 71 change if the two polynomials can have the same degree?

74. How does the answer to Problem 72 change if the two polynomials can have the same degree?

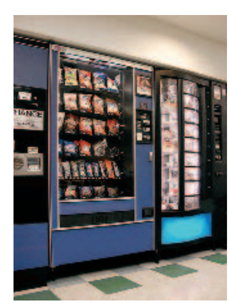
APPLICATIONS

75. GEOMETRY The width of a rectangle is 5 centimeters less than its length. If x represents the length, write an algebraic expression in terms of x that represents the perimeter of the rectangle. Simplify the expression.

76. GEOMETRY The length of a rectangle is 8 meters more than its width. If x represents the width of the rectangle, write an algebraic expression in terms of x that represents its area. Change the expression to a form without parentheses.

★**77.** COIN PROBLEM A parking meter contains nickels, dimes, and quarters. There are 5 fewer dimes than nickels, and 2 more quarters than dimes. If x represents the number of nickels, write an algebraic expression in terms of x that represents the value of all the coins in the meter in cents. Simplify the expression.

★**78.** COIN PROBLEM A vending machine contains dimes and quarters only. There are 4 more dimes than quarters. If x represents the number of quarters, write an algebraic expression in terms of x that represents the value of all the coins in the vending machine in cents. Simplify the expression.



79. PACKAGING A spherical plastic container for designer wristwatches has an inner radius of x centimeters (see the figure). If the plastic shell is 0.3 centimeters thick, write an algebraic expression in terms of x that represents the volume of the plastic used to construct the container. Simplify the expression. [*Recall:* The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.]

80. PACKAGING A cubical container for shipping computer components is formed by coating a metal mold with polystyrene. If the metal mold is a cube with sides x centimeters long and the polystyrene coating is 2 centimeters thick, write an algebraic expression in terms of x that represents the volume of the polystyrene used to construct the container. Simplify the expression. [*Recall:* The volume V of a cube with sides of length t is given by $V = t^3$.]

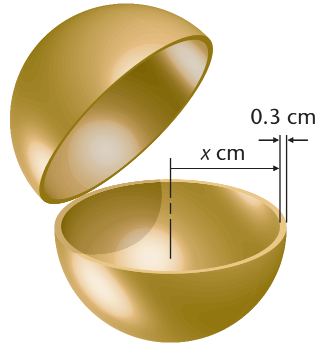


Figure for 79

R-5 Polynomials: Factoring

Factoring—What Does It Mean?

A **factor of a number** is one of two or more numbers whose product is the given number. Similarly, a **factor of an algebraic expression** is one of two or more algebraic expressions whose product is the given algebraic expression. For example,

$$30 = 2 \cdot 3 \cdot 5 \qquad \text{2, 3, and 5 are each factors of 30.}$$

$$x^2 - 4 = (x - 2)(x + 2) \qquad (x - 2) \text{ and } (x + 2) \text{ are each factors of } x^2 - 4.$$

The process of writing a number or algebraic expression as the product of other numbers or algebraic expressions is called **factoring.** We start our discussion of factoring with the positive integers.

An integer such as 30 can be represented in a factored form in many ways. The products

$$6 \cdot 5 \qquad (\tfrac{1}{2})(10)(6) \qquad 15 \cdot 2 \qquad 2 \cdot 3 \cdot 5$$

all yield 30. A particularly useful way of factoring positive integers greater than 1 is in terms of *prime* numbers.

DEFINITION 1 Prime and Composite Numbers

An integer greater than 1 is **prime** if its only positive integer factors are itself and 1. An integer greater than 1 that is not prime is called a **composite number.** The integer 1 is neither prime nor composite.

Examples of prime numbers: 2, 3, 5, 7, 11, 13

Examples of composite numbers: 4, 6, 8, 9, 10, 12

EXPLORE-DISCUSS 1

In the array below, cross out all multiples of 2, except 2 itself. Then cross out all multiples of 3, except 3 itself. Repeat this for each integer in the array that has not yet been crossed out. Describe the set of numbers that remains when this process is completed.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

This process is referred to as the **sieve of Eratosthenes.** (Eratosthenes was a Greek mathematician and astronomer who was a contemporary of Archimedes, circa 200 B.C.)

A composite number is said to be **factored completely** if it is represented as a product of prime factors. The only factoring of 30 given on page 47 that meets this condition is $30 = 2 \cdot 3 \cdot 5$.

EXAMPLE 1 Factoring a Composite Number

Write 60 in completely factored form.

SOLUTION

$$60 = 6 \cdot 10 = 2 \cdot 3 \cdot 2 \cdot 5 = 2^2 \cdot 3 \cdot 5$$

or

$$60 = 5 \cdot 12 = 5 \cdot 4 \cdot 3 = 2^2 \cdot 3 \cdot 5$$

or

$$60 = 2 \cdot 30 = 2 \cdot 2 \cdot 15 = 2^2 \cdot 3 \cdot 5$$

MATCHED PROBLEM 1

Write 180 in completely factored form.

Notice in Example 1 that we end up with the same prime factors for 60 irrespective of how we progress through the factoring process. This illustrates an important property of integers:

› THEOREM 1 The Fundamental Theorem of Arithmetic

Each integer greater than 1 is either prime or can be expressed uniquely, except for the order of factors, as a product of prime factors.

We can also write polynomials in completely factored form. A polynomial such as $2x^2 - x - 6$ can be written in factored form in many ways. The products

$$(2x + 3)(x - 2) \qquad 2(x^2 - \tfrac{1}{2}x - 3) \qquad 2(x + \tfrac{3}{2})(x - 2)$$

all yield $2x^2 - x - 6$. A particularly useful way of factoring polynomials is in terms of prime polynomials.

› DEFINITION 2 Prime Polynomials

A polynomial of degree greater than 0 is said to be **prime** relative to a given set of numbers if: (1) all of its coefficients are from that set of numbers; and (2) it cannot be written as a product of two polynomials (excluding constant polynomials that are factors of 1) having coefficients from that set of numbers.

Relative to the set of integers:

$x^2 - 2$ is prime
$x^2 - 9$ is not prime, since $x^2 - 9 = (x - 3)(x + 3)$

[*Note:* The set of numbers most frequently used in factoring polynomials is the set of integers.]

A nonprime polynomial is said to be **factored completely relative to a given set of numbers** if it is written as a product of prime polynomials relative to that set of numbers.

Our objective in this section is to review some of the standard factoring techniques for polynomials with integer coefficients. In Chapter 3 we will treat in detail the topic of factoring polynomials of higher degree with arbitrary coefficients.

Common Factors and Factoring by Grouping

Example 2 illustrates the use of the distributive properties in factoring.

EXAMPLE 2 Factoring Out Common Factors

Factor out, relative to the integers, all factors common to all terms:

(A) $2x^3y - 8x^2y^2 - 6xy^3$ (B) $2x(3x - 2) - 7(3x - 2)$

SOLUTIONS

(A) $2x^3y - 8x^2y^2 - 6xy^3 = (2xy)x^2 - (2xy)4xy - (2xy)3y^2$ Factor out $2xy$.

$= 2xy(x^2 - 4xy - 3y^2)$

(B) $2x(3x - 2) - 7(3x - 2) = 2x(3x - 2) - 7(3x - 2)$ Factor out $3x - 2$.

$= (2x - 7)(3x - 2)$

MATCHED PROBLEM 2

Factor out, relative to the integers, all factors common to all terms:

(A) $3x^3y - 6x^2y^2 - 3xy^3$ (B) $3y(2y + 5) + 2(2y + 5)$

EXAMPLE 3 Factoring Out Common Factors

Factor completely, relative to the integers:

$$4(2x + 7)(x - 3)^2 + 2(2x + 7)^2(x - 3)$$

SOLUTION

$$4(2x + 7)(x - 3)^2 + 2(2x + 7)^2(x - 3)$$ Factor out $2(2x + 7)(x - 3)$.

$$= 2(2x + 7)(x - 3)[2(x - 3) + (2x + 7)]$$ Distribute, remove parentheses.

$$= 2(2x + 7)(x - 3)(2x - 6 + 2x + 7)$$ Combine like terms.

$$= 2(2x + 7)(x - 3)(4x + 1)$$

MATCHED PROBLEM 3

Factor completely, relative to the integers:

$$4(2x + 5)(3x + 1)^2 + 6(2x + 5)^2(3x + 1)$$

Some polynomials can be factored by first grouping terms to obtain an algebraic expression that looks something like Example 2B. We can then complete the factoring by the method used in that example.

EXAMPLE 4 Factoring by Grouping

Factor completely, relative to the integers, by grouping:

(A) $3x^2 - 6x + 4x - 8$ (B) $wy + wz - 2xy - 2xz$

(C) $3ac + bd - 3ad - bc$

SOLUTIONS

(A) $3x^2 - 6x + 4x - 8$ — Group the first two and last two terms.

$= (3x^2 - 6x) + (4x - 8)$ — Remove common factors from each group.

$= 3x(x - 2) + 4(x - 2)$ — Factor out the common factor $(x - 2)$.

$= (3x + 4)(x - 2)$

(B) $wy + wz - 2xy - 2xz$ — Group the first two and last two terms—be careful of signs.

$= (wy + wz) - (2xy + 2xz)$ — Remove common factors from each group.

$= w(y + z) - 2x(y + z)$ — Factor out the common factor $(y + z)$.

$= (w - 2x)(y + z)$

(C) $3ac + bd - 3ad - bc$

In parts (A) and (B) the polynomials are arranged in such a way that grouping the first two terms and the last two terms leads to common factors. In this problem neither the first two terms nor the last two terms have a common factor. Sometimes rearranging terms will lead to a factoring by grouping. In this case, we interchange the second and fourth terms to obtain a problem comparable to part (B), which can be factored as follows:

$3ac - bc - 3ad + bd = (3ac - bc) - (3ad - bd)$ — Factor out c, d.

$= c(3a - b) - d(3a - b)$ — Factor out $3a - b$.

$= (c - d)(3a - b)$

MATCHED PROBLEM 4

Factor completely, relative to the integers, by grouping:

(A) $2x^2 + 6x + 5x + 15$ (B) $2pr + ps - 6qr - 3qs$

(C) $6wy - xz - 2xy + 3wz$

Factoring Second-Degree Polynomials

We now turn our attention to factoring second-degree polynomials of the form

$$2x^2 - 5x - 3 \qquad \text{and} \qquad 2x^2 + 3xy - 2y^2$$

into the product of two first-degree polynomials with integer coefficients. Examples will illustrate an approach to the problem.

EXAMPLE 5 Factoring Second-Degree Polynomials

Factor each polynomial, if possible, using integer coefficients:

(A) $2x^2 + 3xy - 2y^2$ (B) $x^2 - 3x + 4$ (C) $6x^2 + 5xy - 4y^2$

SOLUTIONS

(A) $2x^2 + 3xy - 2y^2 = (2x + \underset{?}{\uparrow}y)(x - \underset{?}{\uparrow}y)$

Put in what we know. Signs must be opposite. (We can reverse this choice if we get $-3xy$ instead of $+3xy$ for the middle term.)

Now, what are the factors of 2 (the coefficient of y^2)?

2	
$1 \cdot 2$	$(2x + y)(x - 2y) = 2x^2 - 3xy - 2y^2$
$2 \cdot 1$	$(2x + 2y)(x - y) = 2x^2 - 2y^2$

The first choice gives us $-3xy$ for the middle term—close, but not there—so we reverse our choice of signs to obtain

$$2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$$

(B) $x^2 - 3x + 4 = (x - \ \ )(x - \ \ )$

Signs must be the same because the third term is positive and must be negative because the middle term is negative.

4	
$2 \cdot 2$	$(x - 2)(x - 2) = x^2 - 4x + 4$
$1 \cdot 4$	$(x - 1)(x - 4) = x^2 - 5x + 4$
$4 \cdot 1$	$(x - 4)(x - 1) = x^2 - 5x + 4$

No choice produces the middle term; hence $x^2 - 3x + 4$ is not factorable using integer coefficients.

(C) $6x^2 + 5xy - 4y^2 = (\underset{?}{\uparrow}x + \underset{?}{\uparrow}y)(\underset{?}{\uparrow}x - \underset{?}{\uparrow}y)$

The signs must be opposite in the factors, because the third term is negative. We can reverse our choice of signs later if necessary. We now write all factors of 6 and of 4:

6	4
$2 \cdot 3$	$2 \cdot 2$
$3 \cdot 2$	$1 \cdot 4$
$1 \cdot 6$	$4 \cdot 1$
$6 \cdot 1$	

and try each choice on the left with each on the right—a total of 12 combinations that give us the first and last terms in the polynomial $6x^2 + 5xy - 4y^2$. The question is: Does any combination also give us the middle term, $5xy$? After trial and error and, perhaps, some educated guessing among the choices, we find that $3 \cdot 2$ matched with $4 \cdot 1$ gives us the correct middle term. Thus,

$$6x^2 + 5xy - 4y^2 = (3x + 4y)(2x - y)$$

If none of the 24 combinations (including reversing our sign choice) had produced the middle term, then we would conclude that the polynomial is not factorable using integer coefficients.

MATCHED PROBLEM 5

Factor each polynomial, if possible, using integer coefficients:

(A) $x^2 - 8x + 12$ (B) $x^2 + 2x + 5$

(C) $2x^2 + 7xy - 4y^2$ (D) $4x^2 - 15xy - 4y^2$

More Factoring

The special factoring formulas listed here will enable us to factor certain polynomial forms that occur frequently.

SPECIAL FACTORING FORMULAS

1. $u^2 + 2uv + v^2 = (u + v)^2$ **Perfect Square**
2. $u^2 - 2uv + v^2 = (u - v)^2$ **Perfect Square**
3. $u^2 - v^2 = (u - v)(u + v)$ **Difference of Squares**
4. $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$ **Difference of Cubes**
5. $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$ **Sum of Cubes**

The formulas in the box can be established by multiplying the factors on the right.

››› **CAUTION** ›››

Note that we did not list a special factoring formula for the sum of two squares. In general,

$$u^2 + v^2 \neq (au + bv)(cu + dv)$$

for any choice of real number coefficients a, b, c, and d. In Chapter 1 we will see that $u^2 + v^2$ can be factored using complex numbers.

EXAMPLE 6

Using Special Factoring Formulas

Factor completely relative to the integers:

(A) $x^2 + 6xy + 9y^2$ (B) $9x^2 - 4y^2$ (C) $8m^3 - 1$ (D) $x^3 + y^3z^3$

SOLUTIONS

(A) $x^2 + 6xy + 9y^2 = x^2 + 2(x)(3y) + (3y)^2 = (x + 3y)^2$ — Perfect square

(B) $9x^2 - 4y^2 = (3x)^2 - (2y)^2 = (3x - 2y)(3x + 2y)$ — Difference of squares

(C) $8m^3 - 1 = (2m)^3 - 1^3$
$= (2m - 1)[(2m)^2 + (2m)(1) + 1^2]$ — Difference of cubes
Simplify.
$= (2m - 1)(4m^2 + 2m + 1)$

(D) $x^3 + y^3z^3 = x^3 + (yz)^3$ — Sum of cubes
$= (x + yz)(x^2 - xyz + y^2z^2)$

MATCHED PROBLEM 6

Factor completely relative to the integers:

(A) $4m^2 - 12mn + 9n^2$ (B) $x^2 - 16y^2$ (C) $z^3 - 1$ (D) $m^3 + n^3$

››› EXPLORE-DISCUSS 2

(A) Verify the following factor formulas for $u^4 - v^4$:

$$\begin{aligned} u^4 - v^4 &= (u - v)(u + v)(u^2 + v^2) \\ &= (u - v)(u^3 + u^2v + uv^2 + v^3) \end{aligned}$$

(B) Discuss the pattern in the following formulas:

$$\begin{aligned} u^2 - v^2 &= (u - v)(u + v) \\ u^3 - v^3 &= (u - v)(u^2 + uv + v^2) \\ u^4 - v^4 &= (u - v)(u^3 + u^2v + uv^2 + v^3) \end{aligned}$$

(C) Use the pattern you discovered in part (B) to write similar formulas for $u^5 - v^5$ and $u^6 - v^6$. Verify your formulas by multiplication.

We complete this section by considering factoring that involves combinations of the preceding techniques as well as a few additional ones. Generally speaking:

When asked to factor a polynomial, we first take out all factors common to all terms, if they are present, and then proceed to factor until all factors are prime.

EXAMPLE 7 Combining Factoring Techniques

Factor completely relative to the integers:

(A) $18x^3 - 8x$ (B) $x^2 - 6x + 9 - y^2$ (C) $4m^3n - 2m^2n^2 + 2mn^3$

(D) $2t^4 - 16t$ (E) $2y^4 - 5y^2 - 12$

SOLUTIONS

(A) $$\begin{aligned} 18x^3 - 8x &= 2x(9x^2 - 4) \\ &= 2x(3x - 2)(3x + 2) \end{aligned}$$

(B) $x^2 - 6x + 9 - y^2$ — Group the first three terms.

$= (x^2 - 6x + 9) - y^2$ — Factor $x^2 - 6x + 9$.

$= (x - 3)^2 - y^2$ — Difference of squares

$= [(x - 3) - y][(x - 3) + y]$ — Remove parentheses.

$= (x - 3 - y)(x - 3 + y)$

(C) $4m^3n - 2m^2n^2 + 2mn^3 = 2mn(2m^2 - mn + n^2)$

(D) $2t^4 - 16t = 2t(t^3 - 8)$ Difference of cubes
$= 2t(t - 2)(t^2 + 2t + 4)$

(E) $2y^4 - 5y^2 - 12 = (2y^2 + 3)(y^2 - 4)$ Difference of squares
$= (2y^2 + 3)(y - 2)(y + 2)$

MATCHED PROBLEM 7

Factor completely relative to the integers:

(A) $3x^3 - 48x$ (B) $x^2 - y^2 - 4y - 4$
(C) $3u^4 - 3u^3v + 9u^2v^2$ (D) $3m^4 - 24mn^3$
(E) $3x^4 - 5x^2 + 2$

ANSWERS TO MATCHED PROBLEMS

1. $2^2 \cdot 3^2 \cdot 5$ **2.** (A) $3xy(x^2 - 2xy - y^2)$ (B) $(3y + 2)(2y + 5)$
3. $2(2x + 5)(3x + 1)(12x + 17)$
4. (A) $(2x + 5)(x + 3)$ (B) $(p - 3q)(2r + s)$ (C) $(3w - x)(2y + z)$
5. (A) $(x - 2)(x - 6)$ (B) Not factorable using integers (C) $(2x - y)(x + 4y)$
(D) $(4x + y)(x - 4y)$
6. (A) $(2m - 3n)^2$ (B) $(x - 4y)(x + 4y)$ (C) $(z - 1)(z^2 + z + 1)$
(D) $(m + n)(m^2 - mn + n^2)$
7. (A) $3x(x - 4)(x + 4)$ (B) $(x - y - 2)(x + y + 2)$ (C) $3u^2(u^2 - uv - 3v^2)$
(D) $3m(m - 2n)(m^2 + 2mn + 4n^2)$ (E) $(3x^2 - 2)(x - 1)(x + 1)$

R-5 Exercises

In Problems 1–8, factor out, relative to the integers, all factors common to all terms.

1. $6x^4 - 8x^3 - 2x^2$
2. $6m^4 - 9m^3 - 3m^2$
3. $10x^3y + 20x^2y^2 - 15xy^3$
4. $8u^3v - 6u^2v^2 + 4uv^3$
5. $5x(x + 1) - 3(x + 1)$
6. $7m(2m - 3) + 5(2m - 3)$
7. $2w(y - 2z) - x(y - 2z)$
8. $a(3c + d) - 4b(3c + d)$

In Problems 9–16, factor completely, relative to the integers.

9. $x^2 - 2x + 3x - 6$
10. $2y^2 - 6y + 5y - 15$
11. $6m^2 + 10m - 3m - 5$
12. $5x^2 - 40x - x + 8$
13. $2x^2 - 4xy - 3xy + 6y^2$
14. $3a^2 - 12ab - 2ab + 8b^2$
15. $8ac + 3bd - 6bc - 4ad$
16. $3pr - 2qs - qr + 6ps$

In Problems 17–30, factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

17. $2x^2 + x - 3$
18. $3y^2 - 8y - 3$
19. $x^2 + 3x - 8$
20. $u^2 + 4uv - 12v^2$
21. $x^2 + 5xy - 14y^2$
22. $m^2 + m - 20$

23. $4a^2 - 9b^2$

24. $x^2 + 4y^2$

25. $4x^2 - 20x + 25$

26. $a^2b^2 - c^2$

27. $a^2b^2 + c^2$

28. $9x^2 - 4$

29. $4x^2 + 9$

30. $16x^2 - 25$

In Problems 31–46, factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

31. $6x^2 + 48x + 72$

32. $3z^2 - 28z + 48$

33. $2y^3 - 22y^2 + 48y$

34. $2x^4 - 24x^3 + 40x^2$

35. $16x^2y - 8xy + y$

36. $4xy^2 - 12xy + 9x$

37. $6m^2 - mn - 12n^2$

38. $6s^2 + 7st + 2t^2$

39. $x^3y - 9xy^3$

40. $4u^3v - uv^3$

41. $3m^3 - 6m^2 + 15m$

42. $2x^3 - 2x^2 + 8x$

43. $m^3 + n^3$

44. $r^3 - t^3$

45. $8x^3 - 125$

46. $27y^3 + 1$

Problems 47–54 are calculus-related. Factor completely, relative to the integers.

47. $2x(x + 1)^4 + 4x^2(x + 1)^3$

48. $(x - 1)^3 + 3x(x - 1)^2$

49. $6(3x - 5)(2x - 3)^2 + 4(3x - 5)^2(2x - 3)$

50. $2(x - 3)(4x + 7)^2 + 8(x - 3)^2(4x + 7)$

51. $5x^4(9 - x)^4 - 4x^5(9 - x)^3$

52. $3x^4(x - 7)^2 + 4x^3(x - 7)^3$

53. $2(x + 1)(x^2 - 5)^2 + 4x(x + 1)^2(x^2 - 5)$

54. $4(x - 3)^3(x^2 + 2)^3 + 6x(x - 3)^4(x^2 + 2)^2$

In Problems 55–60, factor completely, relative to the integers. In polynomials involving more than three terms, try grouping the terms in various combinations as a first step. If a polynomial is prime relative to the integers, say so.

55. $(a - b)^2 - 4(c - d)^2$

56. $(x + 2)^2 + 9$

57. $2am - 3an + 2bm - 3bn$

58. $15ac - 20ad + 3bc - 4bd$

59. $3x^2 - 2xy - 4y^2$

60. $5u^2 + 4uv - v^2$

In Problems 61–78, factor completely, relative to the integers. In polynomials involving more than three terms, try grouping the terms as a first step. If a polynomial is prime relative to the integers, say so.

61. $x^3 - 3x^2 - 9x + 27$

62. $x^3 - x^2 - x + 1$

63. $a^3 - 2a^2 - a + 2$

64. $t^3 - 2t^2 + t - 2$

65. $4(A + B)^2 - 5(A + B) - 5$

66. $6(x - y)^2 + 23(x - y) - 4$

67. $x^4 + 6x^2 + 8$

68. $x^4 - x^2 - 6$

69. $m^4 - n^4$

70. $y^4 - 3y^2 - 4$

71. $y^4 - 11y^2 + 18$

72. $27a^2 + a^5b^3$

73. $m^2 + 2mn + n^2 - m - n$

74. $y^2 - 2xy + x^2 - y + x$

75. $18a^3 - 8a(x^2 + 8x + 16)$

76. $25(4x^2 - 12xy + 9y^2) - 9a^2b^2$

77. $x^4 + 2x^2 + 1 - x^2$

78. $a^4 + 2a^2b^2 + b^4 - a^2b^2$

79. Find a factor formula for $u^7 - v^7$, where the first factor is $u - v$ and the second factor is a sum of terms of the form au^nv^m. Verify your formula by multiplication.

80. Show that $u^8 - v^8 = (u - v)(u + v)(u^2 + v^2)(u^4 + v^4)$.

81. Use the Sieve of Eratosthenes to find all prime numbers less than 200.

82. If p and $q = p + 2$ are both prime numbers, then p and q are called **twin primes.** Use the results of Problem 81 to find all twin primes less than 200.

83. To show that $\sqrt{2}$ is an irrational number, explain how the assumption that $\sqrt{2}$ is rational leads to a contradiction of Theorem 1, the fundamental theorem of arithmetic, by the following steps:

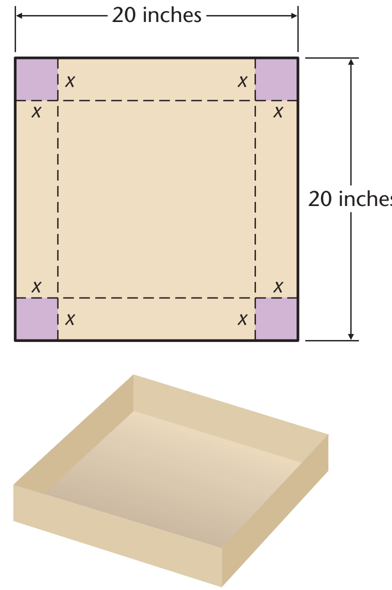
(A) Suppose that $\sqrt{2} = a/b$, where a and b are positive integers, $b \neq 0$. Explain why $a^2 = 2b^2$.

(B) Explain why the prime number 2 appears an even number of times (possibly 0 times) as a factor in the prime factorization of a^2.

(C) Explain why the prime number 2 appears an odd number of times as a factor in the prime factorization of $2b^2$.

(D) Explain why parts (B) and (C) contradict the fundamental theorem of arithmetic.

84. To show that $\sqrt{n}$ is an irrational number unless n is a perfect square, explain how the assumption that $\sqrt{n}$ is rational leads to a contradiction of the fundamental theorem of arithmetic by the following steps:
(A) Assume that n is not a perfect square, that is, does not belong to the sequence 1, 4, 9, 16, 25, ... Explain why some prime number p appears an odd number of times as a factor in the prime factorization of n.
(B) Suppose that $\sqrt{n} = a/b$, where a and b are positive integers, $b \neq 0$. Explain why $a^2 = nb^2$.
(C) Explain why the prime number p appears an even number of times (possibly 0 times) as a factor in the prime factorization of a^2.
(D) Explain why the prime number p appears an odd number of times as a factor in the prime factorization of nb^2.
(E) Explain why parts (C) and (D) contradict the fundamental theorem of arithmetic.



APPLICATIONS

85. CONSTRUCTION A rectangular open-topped box is to be constructed out of 20-inch-square sheets of thin cardboard by cutting x-inch squares out of each corner and bending the sides up as indicated in the figure. Express each of the following quantities as a polynomial in both factored and expanded form.
(A) The area of cardboard after the corners have been removed.
(B) The volume of the box.

86. CONSTRUCTION A rectangular open-topped box is to be constructed out of 9- by 16-inch sheets of thin cardboard by cutting x-inch squares out of each corner and bending the sides up. Express each of the following quantities as a polynomial in both factored and expanded form.
(A) The area of cardboard after the corners have been removed.
(B) The volume of the box.

R-6 Rational Expressions: Basic Operations

- Reducing to Lowest Terms
- Multiplication and Division
- Addition and Subtraction
- Compound Fractions

A quotient of two algebraic expressions, division by 0 excluded, is called a **fractional expression.** If both the numerator and denominator of a fractional expression are polynomials, the fractional expression is called a **rational expression.** Some examples of rational expressions are the following (recall, a nonzero constant is a polynomial of degree 0):

$$\frac{x-2}{2x^2-3x+5} \qquad \frac{1}{x^4-1} \qquad \frac{3}{x} \qquad \frac{x^2+3x-5}{1}$$

In this section we discuss basic operations on rational expressions, including multiplication, division, addition, and subtraction.

Since variables represent real numbers in the rational expressions we are going to consider, the properties of real number fractions summarized in Section R-1 play a central role in much of the work that we will do.

Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded.

Reducing to Lowest Terms

We start this discussion by restating the **fundamental property of fractions** (from Theorem 3 in Section R-1):

FUNDAMENTAL PROPERTY OF FRACTIONS

If a, b, and k are real numbers with $b, k \neq 0$, then

$$\frac{ka}{kb} = \frac{a}{b} \qquad \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4} \qquad \frac{(x-3)2}{(x-3)x} = \frac{2}{x}$$

$x \neq 0, x \neq 3$

Using this property from left to right to eliminate all common factors from the numerator and the denominator of a given fraction is referred to as **reducing a fraction to lowest terms.** We are actually dividing the numerator and denominator by the same nonzero common factor.

Using the property from right to left—that is, multiplying the numerator and the denominator by the same nonzero factor—is referred to as **raising a fraction to higher terms.** We will use the property in both directions in the material that follows.

We say that a rational expression is **reduced to lowest terms** if the numerator and denominator do not have any factors in common. Unless stated to the contrary, factors will be relative to the integers.

EXAMPLE 1 Reducing Rational Expressions

Reduce each rational expression to lowest terms.

(A) $$\frac{x^2 - 6x + 9}{x^2 - 9} = \frac{(x-3)^2}{(x-3)(x+3)}$$

Factor numerator and denominator completely. Divide numerator and denominator by $(x - 3)$; this is a valid operation as long as $x \neq 3$.

$$= \frac{x-3}{x+3}$$

(B) $\dfrac{x^3 - 1}{x^2 - 1} = \dfrac{\overset{1}{\cancel{(x - 1)}}(x^2 + x + 1)}{\underset{1}{\cancel{(x - 1)}}(x + 1)}$

$= \dfrac{x^2 + x + 1}{x + 1}$

Dividing numerator and denominator by $(x - 1)$ can be indicated by drawing lines through both $(x - 1)$'s and writing the resulting quotients, 1's.

$x \neq -1$ and $x \neq 1$

MATCHED PROBLEM 1

Reduce each rational expression to lowest terms.

(A) $\dfrac{6x^2 + x - 2}{2x^2 + x - 1}$ (B) $\dfrac{x^4 - 8x}{3x^3 - 2x^2 - 8x}$

››› CAUTION ›››

Remember to always factor the numerator and denominator first, then divide out any *common factors.* Do not indiscriminately eliminate *terms* that appear in both the numerator and the denominator. For example,

$$\frac{2x^3 + y^2}{y^2} \neq \frac{2x^3 + \overset{1}{\cancel{y^2}}}{\underset{1}{\cancel{y^2}}}$$

$$\frac{2x^3 + y^2}{y^2} \neq 2x^3 + 1$$

Since the term y^2 is not a factor of the numerator, it cannot be eliminated. In fact, $(2x^3 + y^2)/y^2$ is already reduced to lowest terms.

Multiplication and Division

Since we are restricting variable replacements to real numbers, multiplication and division of rational expressions follow the rules for multiplying and dividing real number fractions (Theorem 3 in Section R-1).

MULTIPLICATION AND DIVISION

If a, b, c, and d are real numbers with $b, d \neq 0$, then:

1. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ $\dfrac{2}{3} \cdot \dfrac{x}{x - 1} = \dfrac{2x}{3(x - 1)}$

2. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$ $c \neq 0$ $\dfrac{2}{3} \div \dfrac{x}{x - 1} = \dfrac{2}{3} \cdot \dfrac{x - 1}{x}$

››› EXPLORE-DISCUSS 1

Write a verbal description of the process of multiplying two fractions. Do the same for the quotient of two fractions.

EXAMPLE 2 Multiplying and Dividing Rational Expressions

Perform the indicated operations and reduce to lowest terms.

(A) $$\frac{10x^3y}{3xy+9y}\cdot\frac{x^2-9}{4x^2-12x}=\frac{\overset{5x^2}{\cancel{10x^3y}}}{\underset{3\cdot 1}{\cancel{3y(x+3)}}}\cdot\frac{\overset{1\cdot 1}{\cancel{(x-3)(x+3)}}}{\underset{2\cdot 1}{\cancel{4x(x-3)}}}=\frac{5x^2}{6}$$

Factor numerators and denominators; then divide any numerator and any denominator with a like common factor.

(B) $$\frac{4-2x}{4}\div(x-2)=\frac{\overset{1}{\cancel{2}}(2-x)}{\underset{2}{\cancel{4}}}\cdot\frac{1}{x-2}$$

$x-2$ is the same as $\frac{x-2}{1}$.

$$=\frac{2-x}{2(x-2)}=\frac{-\overset{-1}{\cancel{(x-2)}}}{2\underset{1}{\cancel{(x-2)}}}$$

$b-a=-(a-b)$, a useful change in some problems.

$$=-\frac{1}{2}$$

(C) $$\frac{2x^3-2x^2y+2xy^2}{x^3y-xy^3}\div\frac{x^3+y^3}{x^2+2xy+y^2}$$

$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$

$$=\frac{\overset{2}{\cancel{2x}}\overset{1}{\cancel{(x^2-xy+y^2)}}}{\underset{y}{\cancel{xy}}\underset{1}{\cancel{(x+y)}}(x-y)}\cdot\frac{\overset{1}{\cancel{(x+y)}}{}^2}{\underset{1}{\cancel{(x+y)}}\underset{1}{\cancel{(x^2-xy+y^2)}}}$$

Divide out common factors.

$$=\frac{2}{y(x-y)}$$

MATCHED PROBLEM 2

Perform the indicated operations and reduce to lowest terms.

(A) $\frac{12x^2y^3}{2xy^2+6xy}\cdot\frac{y^2+6y+9}{3y^3+9y^2}$ (B) $(4-x)\div\frac{x^2-16}{5}$

(C) $\frac{m^3+n^3}{2m^2+mn-n^2}\div\frac{m^3n-m^2n^2+mn^3}{2m^3n^2-m^2n^3}$

Addition and Subtraction

Again, because we are restricting variable replacements to real numbers, addition and subtraction of rational expressions follow the rules for adding and subtracting real number fractions (Theorem 3 in Section R-1).

ADDITION AND SUBTRACTION

For a, b, and c real numbers with $b \neq 0$:

1. $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a + c}{b}$ $\quad \dfrac{x}{x - 3} + \dfrac{2}{x - 3} = \dfrac{x + 2}{x - 3}$
2. $\dfrac{a}{b} - \dfrac{c}{b} = \dfrac{a - c}{b}$ $\quad \dfrac{x}{2xy^2} - \dfrac{x - 4}{2xy^2} = \dfrac{x - (x - 4)}{2xy^2}$

Thus, we add rational expressions with the same denominators by adding or subtracting their numerators and placing the result over the common denominator. If the denominators are not the same, we raise the fractions to higher terms, using the fundamental property of fractions to obtain common denominators, and then proceed as described.

Even though any common denominator will do, our work will be simplified if the least common denominator (LCD) is used. Often, the LCD is obvious, but if it is not, the steps in the box describe how to find it.

THE LEAST COMMON DENOMINATOR (LCD)

The LCD of two or more rational expressions is found as follows:

1. Factor each denominator completely.
2. Identify each different prime factor from all the denominators.
3. Form a product using each different factor to the highest power that occurs in any one denominator. This product is the LCD.

EXAMPLE 3 Adding and Subtracting Rational Expressions

Combine into a single fraction and reduce to lowest terms.

(A) $\dfrac{3}{10} + \dfrac{5}{6} - \dfrac{11}{45}$ (B) $\dfrac{4}{9x} - \dfrac{5x}{6y^2} + 1$

(C) $\dfrac{x + 3}{x^2 - 6x + 9} - \dfrac{x + 2}{x^2 - 9} - \dfrac{5}{3 - x}$

SOLUTIONS

(A) To find the LCD, factor each denominator completely:

$$\left.\begin{aligned} 10 &= 2 \cdot 5 \\ 6 &= 2 \cdot 3 \\ 45 &= 3^2 \cdot 5 \end{aligned}\right\} \text{LCD} = 2 \cdot 3^2 \cdot 5 = 90$$

Now use the fundamental property of fractions to make each denominator 90:

$$\frac{3}{10} + \frac{5}{6} - \frac{11}{45} = \frac{9 \cdot 3}{9 \cdot 10} + \frac{15 \cdot 5}{15 \cdot 6} - \frac{2 \cdot 11}{2 \cdot 45}$$ Multiply.

$$= \frac{27}{90} + \frac{75}{90} - \frac{22}{90}$$ Combine into a single fraction.

$$= \frac{27 + 75 - 22}{90} = \frac{80}{90} = \frac{8}{9}$$

(B) $$\left.\begin{aligned} 9x &= 3^2x \\ 6y^2 &= 2 \cdot 3y^2 \end{aligned}\right\} \text{LCD} = 2 \cdot 3^2xy^2 = 18xy^2$$

$$\frac{4}{9x} - \frac{5x}{6y^2} + 1 = \frac{2y^2 \cdot 4}{2y^2 \cdot 9x} - \frac{3x \cdot 5x}{3x \cdot 6y^2} + \frac{18xy^2}{18xy^2}$$ Multiply, combine.

$$= \frac{8y^2 - 15x^2 + 18xy^2}{18xy^2}$$

(C) $$\frac{x + 3}{x^2 - 6x + 9} - \frac{x + 2}{x^2 - 9} - \frac{5}{3 - x} = \frac{x + 3}{(x - 3)^2} - \frac{x + 2}{(x - 3)(x + 3)} + \frac{5}{x - 3}$$

Note: $-\frac{5}{3 - x} = -\frac{5}{-(x - 3)} = \frac{5}{x - 3}$ We have again used the fact that $a - b = -(b - a)$.

The LCD $= (x - 3)^2(x + 3)$. Thus,

$$\frac{(x + 3)^2}{(x - 3)^2(x + 3)} - \frac{(x - 3)(x + 2)}{(x - 3)^2(x + 3)} + \frac{5(x - 3)(x + 3)}{(x - 3)^2(x + 3)}$$ Expand numerators.

$$= \frac{(x^2 + 6x + 9) - (x^2 - x - 6) + 5(x^2 - 9)}{(x - 3)^2(x + 3)}$$ Be careful of sign errors here.

$$= \frac{x^2 + 6x + 9 - x^2 + x + 6 + 5x^2 - 45}{(x - 3)^2(x + 3)}$$ Combine like terms.

$$= \frac{5x^2 + 7x - 30}{(x - 3)^2(x + 3)}$$

MATCHED PROBLEM 3

Combine into a single fraction and reduce to lowest terms.

(A) $\frac{5}{28} - \frac{1}{10} + \frac{6}{35}$ (B) $\frac{1}{4x^2} - \frac{2x + 1}{3x^3} + \frac{3}{12x}$

(C) $\frac{y - 3}{y^2 - 4} - \frac{y + 2}{y^2 - 4y + 4} - \frac{2}{2 - y}$

EXPLORE-DISCUSS 2

What is the value of $\frac{\frac{16}{4}}{2}$?

What is the result of entering $16 \div 4 \div 2$ on a calculator?

What is the difference between $16 \div (4 \div 2)$ and $(16 \div 4) \div 2$?

How could you use fraction bars to distinguish between these two cases when writing $\frac{16}{\frac{4}{2}}$?

Compound Fractions

A fractional expression with fractions in its numerator, denominator, or both is called a **compound fraction.** It is often necessary to represent a compound fraction as a **simple fraction**—that is (in all cases we will consider), as the quotient of two polynomials. The process does not involve any new concepts. It is a matter of applying old concepts and processes in the right sequence. We will illustrate two approaches to the problem, each with its own merits, depending on the particular problem under consideration.

EXAMPLE 4 **Simplifying Compound Fractions**

Express as a simple fraction reduced to lowest terms:

$$\frac{\frac{2}{x} - 1}{\frac{4}{x^2} - 1}$$

SOLUTION

Method 1. Multiply the numerator and denominator by the LCD of all fractions in the numerator and denominator—in this case, x^2. (We are multiplying by $1 = x^2/x^2$.)

$$\frac{x^2\left(\frac{2}{x} - 1\right)}{x^2\left(\frac{4}{x^2} - 1\right)} = \frac{x^2\frac{2}{x} - x^2}{x^2\frac{4}{x^2} - x^2} = \frac{2x - x^2}{4 - x^2} = \frac{x(2 - x)}{(2 + x)(2 - x)}$$

$$= \frac{x}{2 + x}$$

Method 2. Write the numerator and denominator as single fractions. Then treat as a quotient.

$$\frac{\frac{2}{x} - 1}{\frac{4}{x^2} - 1} = \frac{\frac{2 - x}{x}}{\frac{4 - x^2}{x^2}} = \frac{2 - x}{x} \div \frac{4 - x^2}{x^2} = \frac{2 - x}{x} \cdot \frac{x^2}{(2 - x)(2 + x)}$$

$$= \frac{x}{2 + x}$$

MATCHED PROBLEM 4

Express as a simple fraction reduced to lowest terms. Use the two methods described in Example 4.

$$\frac{1 + \frac{1}{x}}{x - \frac{1}{x}}$$

ANSWERS TO MATCHED PROBLEMS

1. (A) $\frac{3x + 2}{x + 1}$ (B) $\frac{x^2 + 2x + 4}{3x + 4}$

2. (A) $2x$ (B) $\frac{-5}{x + 4}$ (C) mn

3. (A) $\frac{1}{4}$ (B) $\frac{3x^2 - 5x - 4}{12x^3}$ (C) $\frac{2y^2 - 9y - 6}{(y - 2)^2(y + 2)}$

4. $\frac{1}{x - 1}$

R-6 Exercises

In Problems 1–8, reduce each rational expression to lowest terms.

1. $\frac{42}{105}$

2. $\frac{360}{216}$

3. $\frac{x + 1}{x^2 + 3x + 2}$

4. $\frac{x^2 - 2x - 24}{x - 6}$

5. $\frac{x^2 - 9}{x^2 + 3x - 18}$

6. $\frac{x^2 + 9x + 20}{x^2 - 16}$

7. $\frac{3x^2y^3}{x^4y}$

8. $\frac{2a^2b^4c^6}{6a^5b^3c}$

In Problems 9–28, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

9. $\frac{5}{6} - \frac{11}{15}$

10. $\frac{7}{10} - \frac{19}{25}$

11. $\left(\frac{b^2}{2a} \div \frac{b}{a^2}\right) \cdot \frac{a}{3b}$

12. $\frac{b^2}{2a} \div \left(\frac{b}{a^2} \cdot \frac{a}{3b}\right)$

13. $\frac{1}{n} - \frac{1}{m}$

14. $\frac{m}{n} - \frac{n}{m}$

15. $\frac{x^2 - 1}{x + 2} \div \frac{x + 1}{x^2 - 4}$

16. $\frac{x^2 - 9}{x^2 - 1} \div \frac{x - 3}{x - 1}$

17. $\frac{1}{c} + \frac{1}{b} + \frac{1}{a}$

18. $\frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$

19. $\frac{2a - b}{a^2 - b^2} - \frac{2a + 3b}{a^2 + 2ab + b^2}$

20. $\frac{x + 2}{x^2 - 1} - \frac{x - 2}{(x - 1)^2}$

21. $m + 2 - \frac{m - 2}{m - 1}$

22. $\frac{x + 1}{x - 1} + x$

23. $\frac{3}{x - 2} - \frac{2}{2 - x}$

24. $\frac{1}{a - 3} - \frac{2}{3 - a}$

25. $\frac{3}{y + 2} + \frac{2}{y - 2} - \frac{4y}{y^2 - 4}$

26. $\frac{4x}{x^2 - y^2} + \frac{3}{x + y} - \frac{2}{x - y}$

27. $\frac{\frac{x^2}{y^2} - 1}{\frac{x}{y} + 1}$

28. $\frac{\frac{4}{x} - x}{\frac{2}{x} - 1}$

Problems 29–34 are calculus-related. Reduce each fraction to lowest terms.

29. $\frac{6x^3(x^2 + 2)^2 - 2x(x^2 + 2)^3}{x^4}$

30. $\frac{4x^4(x^2 + 3) - 3x^2(x^2 + 3)^2}{x^6}$

31. $\frac{2x(1 - 3x)^3 + 9x^2(1 - 3x)^2}{(1 - 3x)^6}$

32. $\frac{2x(2x + 3)^4 - 8x^2(2x + 3)^3}{(2x + 3)^8}$

33. $\frac{-2x(x + 4)^3 - 3(3 - x^2)(x + 4)^2}{(x + 4)^6}$

34. $\frac{3x^2(x + 1)^3 - 3(x^3 + 4)(x + 1)^2}{(x + 1)^6}$

In Problems 35–48, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

35. $\dfrac{y}{y^2 - 2y - 8} - \dfrac{2}{y^2 - 5y + 4} + \dfrac{1}{y^2 + y - 2}$

36. $\dfrac{x}{x^2 - 9x + 18} + \dfrac{x - 8}{x - 6} + \dfrac{x + 4}{x - 3}$

37. $\dfrac{16 - m^2}{m^2 + 3m - 4} \cdot \dfrac{m - 1}{m - 4}$

38. $\dfrac{x + 1}{x(1 - x)} \cdot \dfrac{x^2 - 2x + 1}{x^2 - 1}$

39. $\dfrac{x + 7}{ax - bx} + \dfrac{y + 9}{by - ay}$

40. $\dfrac{c + 2}{5c - 5} - \dfrac{c - 2}{3c - 3} + \dfrac{c}{1 - c}$

41. $\dfrac{x^2 - 16}{2x^2 + 10x + 8} \div \dfrac{x^2 - 13x + 36}{x^3 + 1}$

42. $\left(\dfrac{x^3 - y^3}{y^3} \cdot \dfrac{y}{x - y}\right) \div \dfrac{x^2 + xy + y^2}{y^2}$

43. $\dfrac{x^2 - xy}{xy + y^2} \div \left(\dfrac{x^2 - y^2}{x^2 + 2xy + y^2} \div \dfrac{x^2 - 2xy + y^2}{x^2y + xy^2}\right)$

44. $\left(\dfrac{x^2 - xy}{xy + y^2} \div \dfrac{x^2 - y^2}{x^2 + 2xy + y^2}\right) \div \dfrac{x^2 - 2xy + y^2}{x^2y + xy^2}$

45. $\left(\dfrac{x}{x^2 - 16} - \dfrac{1}{x + 4}\right) \div \dfrac{4}{x + 4}$

46. $\left(\dfrac{3}{x - 2} - \dfrac{1}{x + 1}\right) \div \dfrac{x + 4}{x - 2}$

47. $\dfrac{1 + \dfrac{2}{x} - \dfrac{15}{x^2}}{1 + \dfrac{4}{x} - \dfrac{5}{x^2}}$

48. $\dfrac{\dfrac{x}{y} - 2 + \dfrac{y}{x}}{\dfrac{x}{y} - \dfrac{y}{x}}$

Problems 49–52 are calculus-related. Perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

49. $\dfrac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h}$

50. $\dfrac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}$

51. $\dfrac{\dfrac{(x + h)^2}{x + h + 2} - \dfrac{x^2}{x + 2}}{h}$

52. $\dfrac{\dfrac{2x + 2h + 3}{x + h} - \dfrac{2x + 3}{x}}{h}$

In Problems 53–60, imagine that the indicated "solutions" were given to you by a student whom you were tutoring in this class.
(A) Is the solution correct? If the solution is incorrect, explain what is wrong and how it can be corrected.
(B) Show a correct solution for each incorrect solution.

53. $\dfrac{x^2 + 5x + 4}{x + 4} = \dfrac{x^2 + 5x}{x} = x + 5$

54. $\dfrac{x^2 - 2x - 3}{x - 3} = \dfrac{x^2 - 2x}{x} = x - 2$

55. $\dfrac{(x + h)^2 - x^2}{h} = (x + 1)^2 - x^2 = 2x + 1$

56. $\dfrac{(x + h)^3 - x^3}{h} = (x + 1)^3 - x^3 = 3x^2 + 3x + 1$

57. $\dfrac{x^2 - 2x}{x^2 - x - 2} + x - 2 = \dfrac{x^2 - 2x + x - 2}{x^2 - x - 2} = 1$

58. $\dfrac{2}{x - 1} - \dfrac{x + 3}{x^2 - 1} = \dfrac{2x + 2 - x - 3}{x^2 - 1} = \dfrac{1}{x + 1}$

59. $\dfrac{2x^2}{x^2 - 4} - \dfrac{x}{x - 2} = \dfrac{2x^2 - x^2 - 2x}{x^2 - 4} = \dfrac{x}{x + 2}$

60. $x + \dfrac{x - 2}{x^2 - 3x + 2} = \dfrac{x + x - 2}{x^2 - 3x + 2} = \dfrac{2}{x - 2}$

In Problems 61–64, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

61. $\dfrac{y - \dfrac{y^2}{y - x}}{1 + \dfrac{x^2}{y^2 - x^2}}$

62. $\dfrac{\dfrac{s^2}{s - t} - s}{\dfrac{t^2}{s - t} + t}$

63. $2 - \dfrac{1}{1 - \dfrac{2}{a + 2}}$

64. $1 - \dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{x}}}$

In Problems 65 and 66, a, b, c, and d represent real numbers.

65. (A) Prove that d/c is the multiplicative inverse of c/d $(c, d \neq 0)$.
(B) Use part (A) to prove that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \qquad b, c, d \neq 0$$

66. Prove that

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \qquad b \neq 0$$

CHAPTER R Review

R-1 Algebra and Real Numbers

A **set** is a collection of objects called **elements** or **members** of the set. Sets are usually described by **listing** the elements or by stating a **rule** that determines the elements. A set may be **finite** or **infinite.** A set with no elements is called the **empty set** or the **null set** and is denoted $\varnothing$. A **variable** is a symbol that represents unspecified elements from a **replacement set.** A **constant** is a symbol for a single object. If each element of set A is also in set B, we say A is a **subset** of B and write $A \subset B$. If two sets A and B have exactly the same elements, we say that A and B are **equal** and write $A = B$.

The **rational numbers** are numbers that can be written in the form a/b, where a and b are integers and $b \neq 0$. A real number can be approximated to any desired precision by rational numbers. Consequently, arithmetic operations on the rational numbers can be extended to operations on real numbers. These operations satisfy **basic real number properties,** including **associative properties:** $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$; **commutative properties:** $x + y = y + x$ and $xy = yx$; **identities:** $0 + x = x + 0 = x$ and $(1)x = x(1) = x$; **inverses:** $-x$ is the **additive inverse** of x and, if $x \neq 0$, x^{-1} is the **multiplicative inverse** of x; and **distributive property:** $x(y + z) = xy + xz$. **Subtraction** is defined by $a - b = a + (-b)$ and **division** by $a/b = ab^{-1}$. Division by 0 is never allowed. Additional properties include **properties of negatives:**

1. $-(-a) = a$

2. $(-a)b = -(ab) = a(-b) = -ab$

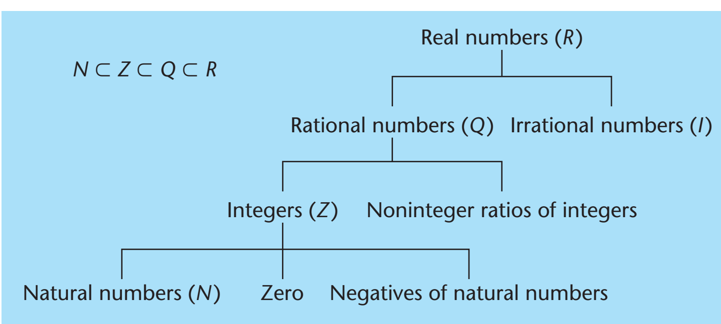
3. $(-a)(-b) = ab$

4. $(-1)a = -a$

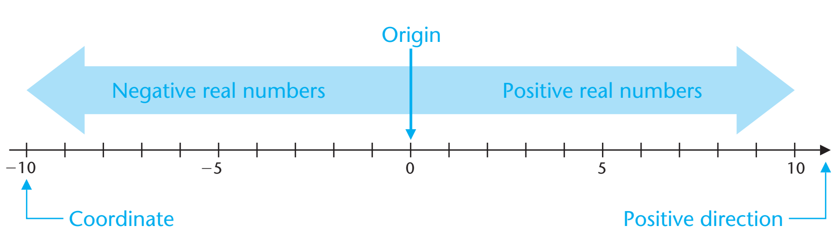
5. $\dfrac{-a}{b} = -\dfrac{a}{b} = \dfrac{a}{-b} \qquad b \neq 0$

6. $\dfrac{-a}{-b} = -\dfrac{-a}{b} = -\dfrac{a}{-b} = \dfrac{a}{b} \qquad b \neq 0$

Real numbers:



Real number line:



zero properties:

1. $a \cdot 0 = 0$

2. $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.

and **fraction properties** (division by 0 excluded):

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$

2. $\frac{ka}{kb} = \frac{a}{b}$

3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

4. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

5. $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

6. $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$

7. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

R-2 Integer Exponents

$a^n = a \cdot a \cdot \cdots \cdot a$ (n factors of a) for n a positive integer, $a^0 = 1$ $(a \neq 0)$, and $a^n = 1/a^{-n}$ for n a negative integer $(a \neq 0)$. 0^0 is not defined.

Properties of integer exponents (division by 0 excluded):

1. $a^m a^n = a^{m+n}$

2. $(a^n)^m = a^{mn}$

3. $(ab)^m = a^m b^m$

4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

5. $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$

Further exponent properties (division by 0 excluded):

1. $(a^m b^n)^p = a^{pm} b^{pn}$

2. $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{pm}}{b^{pn}}$

3. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

4. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

For n a natural number and a and b real numbers:

$$a \text{ is an } n\text{th root of } b \text{ if } a^n = b$$

The **principal *n*th root** of b is denoted by $b^{1/n}$. If n is odd, b has one real nth root which is the principal nth root. If n is even and $b > 0$, b has two real nth roots and the positive nth root is the principal nth root. If n is even and $b < 0$, b has no real nth roots.

Rational number exponents (even roots of negative numbers excluded):

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n} \quad \text{and} \quad b^{-m/n} = \frac{1}{b^{m/n}}$$

Scientific notation:

$$a \times 10^n \qquad 1 \le a < 10$$

n an integer, a in decimal form.

R-3 Radicals

An ***n*th root radical** is defined by $\sqrt[n]{b} = b^{1/n}$, where $b^{1/n}$ is the **principal *n*th root of *b*,** $\sqrt{\ }$ is a **radical,** n is the **index,** and b is the **radicand.** Rational exponents and radicals are related by

$$b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m} = (b^{1/n})^m = (\sqrt[n]{b})^m$$

Properties of radicals $(x > 0, y > 0)$:

1. $\sqrt[n]{x^n} = x$

2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

3. $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

A radical is in **simplified form** if:

1. No radicand contains a factor to a power greater than or equal to the index of the radical.

2. No power of the radicand and the index of the radical have a common factor other than 1.

3. No radical appears in a denominator.

4. No fraction appears within a radical.

Algebraic fractions containing radicals are **rationalized** by multiplying numerator and denominator by a **rationalizing factor** often determined by using a special product formula.

R-4 Polynomials: Basic Operations

An **algebraic expression** is formed by using constants and variables and the operations of addition, subtraction, multiplication, division, raising to powers, and taking roots. A **polynomial** is an algebraic expression formed by adding and subtracting constants and terms of the form ax^n (one variable), ax^ny^m (two variables), and so on. The **degree of a term** is the sum of the powers of all variables in the term, and the **degree of a polynomial** is the degree of the nonzero term with highest degree in the polynomial. Polynomials with one, two, or three terms are called **monomials, binomials,** and **trinomials,** respectively. **Like terms** have exactly the same variable factors to the same powers and can be combined by adding their **coefficients.** Polynomials can be *added, subtracted,* and *multiplied* by repeatedly applying the distributive property and combining like terms. **Special products** of binomials are:

1. $(a - b)(a + b) = a^2 - b^2$

2. $(a - b)^2 = a^2 - 2ab + b^2$

3. $(a + b)^2 = a^2 + 2ab + b^2$

R-5 Polynomials: Factoring

A number or algebraic expression is **factored** if it is expressed as a product of other numbers or algebraic expressions, which are called **factors.** An integer greater than 1 is a **prime number** if its only positive integer factors are itself and 1, and a **composite number** otherwise. Each composite number can be **factored uniquely into a product of prime numbers.** A polynomial is **prime** relative to a given set of numbers (usually the set of integers) if (1) all its coefficients are from that set of numbers, and (2) it cannot be written as a product of two polynomials of positive degree having coefficients from that set of numbers. A nonprime polynomial is **factored completely relative to a given set of numbers** if it is written as a product of prime polynomials relative to that set of numbers. *Common factors* can be factored out by applying the distributive properties. *Grouping* can be used to identify common factors. Second-degree polynomials can be factored by trial and error. The following special factoring formulas are useful:

1. $u^2 + 2uv + v^2 = (u + v)^2$	Perfect Square
2. $u^2 - 2uv + v^2 = (u - v)^2$	Perfect Square
3. $u^2 - v^2 = (u - v)(u + v)$	Difference of Squares
4. $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	Difference of Cubes
5. $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	Sum of Cubes

There is no factoring formula relative to the real numbers for $u^2 + v^2$.

R-6 Rational Expressions: Basic Operations

A **fractional expression** is the ratio of two algebraic expressions, and a **rational expression** is the ratio of two polynomials. The rules for adding, subtracting, multiplying, and dividing real number fractions (see Section R-1 in this review) all extend to fractional expressions with the understanding that **variables are always restricted to exclude division by zero.** Fractions can be **reduced to lowest terms** or **raised to higher terms** by using the **fundamental property of fractions:**

$$\frac{ka}{kb} = \frac{a}{b} \qquad \text{with } b, k \neq 0$$

A rational expression is **reduced to lowest terms** if the numerator and denominator do not have any factors in common relative to the integers. The **least common denominator** (LCD) is useful for adding and subtracting fractions with different denominators and for reducing **compound fractions** to **simple fractions.**

CHAPTER R Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. For $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4\}$, and $C = \{4, 1, 2\}$, indicate true (T) or false (F):
(A) $3 \in A$ (B) $5 \notin C$ (C) $B \in A$
(D) $B \subset A$ (E) $B \neq C$ (F) $A \subset B$

2. Replace each question mark with an appropriate expression that will illustrate the use of the indicated real number property:
(A) Commutative $(\cdot)$: $x(y + z) = ?$
(B) Associative $(+)$: $2 + (x + y) = ?$
(C) Distributive: $(2 + 3)x = ?$

In Problems 3–6, perform the indicated operations and express the result in the form a/b, where a and b are integers.

3. $\left(\frac{2}{3} \cdot \frac{15}{14}\right)^{-1}$ **4.** $\frac{9}{10} + \frac{5}{12}$

5. $-\left(\frac{8}{7} + 2^{-1}\right)$

6. $(4^{-1}9^{-1})^{-1}$

Problems 7–12 refer to the following polynomials: (a) $x^3 + 2$ and (b) $x^5 - x^3 + 1$.

7. What is the degree of (a)?

8. What is the degree of (b)?

9. What is the degree of the sum of (a) and (b)?

10. What is the degree of the product of (a) and (b)?

11. Multiply (a) and (b).

12. Add (a) and (b).

In Problems 13–18, evaluate each expression that results in a rational number.

13. $289^{1/2}$

14. $216^{1/3}$

15. $8^{-2/3}$

16. $(-64)^{5/3}$

17. $\left(\frac{9}{16}\right)^{-1/2}$

18. $(121^{1/2} + 25^{1/2})^{-3/4}$

In Problems 19–22, perform the indicated operations and simplify.

19. $5x^2 - 3x[4 - 3(x - 2)]$

20. $(3m - 5n)(3m + 5n)$

21. $(2x + y)(3x - 4y)$

22. $(2a - 3b)^2$

In Problems 23–25, write each polynomial in a completely factored form relative to the integers. If the polynomial is prime relative to the integers, say so.

23. $9x^2 - 12x + 4$

24. $t^2 - 4t - 6$

25. $6n^3 - 9n^2 - 15n$

In Problems 26–29, perform the indicated operations and reduce to lowest terms. Represent all compound fractions as simple fractions reduced to lowest terms.

26. $\frac{2}{5b} - \frac{4}{3a^3} - \frac{1}{6a^2b^2}$

27. $\frac{3x}{3x^2 - 12x} + \frac{1}{6x}$

28. $\frac{y - 2}{y^2 - 4y + 4} \div \frac{y^2 + 2y}{y^2 + 4y + 4}$

29. $\dfrac{u - \frac{1}{u}}{1 - \frac{1}{u^2}}$

Simplify Problems 30–35, and write answers using positive exponents only. All variables represent positive real numbers.

30. $6(xy^3)^5$

31. $\frac{9u^8v^6}{3u^4v^8}$

32. $(2 \times 10^5)(3 \times 10^{-3})$

33. $(x^{-3}y^2)^{-2}$

34. $u^{5/3}u^{2/3}$

35. $(9a^4b^{-2})^{1/2}$

36. Change to radical form: $3x^{2/5}$

37. Change to rational exponent form: $-3\sqrt[3]{(xy)^2}$

Simplify Problems 38–42, and express answers in simplified form. All variables represent positive real numbers.

38. $3x\sqrt[3]{x^5y^4}$

39. $\sqrt{2x^2y^5}\sqrt{18x^3y^2}$

40. $\frac{6ab}{\sqrt{3a}}$

41. $\frac{\sqrt{5}}{3 - \sqrt{5}}$

42. $\sqrt[8]{y^6}$

43. Write using the listing method:

$$\{x \mid x \text{ is an odd integer between } -4 \text{ and } 2\}$$

In Problems 44–49, each statement illustrates the use of one of the following real number properties or definitions. Indicate which one.

Commutative (+)	*Identity (+)*
Commutative (·)	*Identity (·)*
Division	*Associative (+)*
Inverse (+)	*Associative (·)*
Inverse (·)	*Zero*
Distributive	*Subtraction*
Negatives	

44. $(-3) - (-2) = (-3) + [-(-2)]$

45. $3y + (2x + 5) = (2x + 5) + 3y$

46. $(2x + 3)(3x + 5) = (2x + 3)3x + (2x + 3)5$

47. $3 \cdot (5x) = (3 \cdot 5)x$

48. $\frac{a}{-(b - c)} = -\frac{a}{b - c}$

49. $3xy + 0 = 3xy$

50. Indicate true (T) or false (F):
(A) An integer is a rational number and a real number.
(B) An irrational number has a repeating decimal representation.

51. Give an example of an integer that is not a natural number.

52. Given the algebraic expressions:
(a) $2x^2 - 3x + 5$ (b) $x^2 - \sqrt{x - 3}$
(c) $x^{-3} + x^{-2} - 3x^{-1}$ (d) $x^2 - 3xy - y^2$
(A) Identify all second-degree polynomials.
(B) Identify all third-degree polynomials.

In Problems 53–57, perform the indicated operations and simplify.

53. $(2x - y)(2x + y) - (2x - y)^2$

54. $(m^2 + 2mn - n^2)(m^2 - 2mn - n^2)$

55. $5(x + h)^2 - 7(x + h) - (5x^2 - 7x)$

56. $-2x\{(x^2 + 2)(x - 3) - x[x - x(3 - x)]\}$

57. $(x - 2y)^3$

In Problems 58–64, write in a completely factored form relative to the integers.

58. $(4x - y)^2 - 9x^2$

59. $2x^2 + 4xy - 5y^2$

60. $6x^3y + 12x^2y^2 - 15xy^3$

61. $(y - b)^2 - y + b$

62. $3x^3 + 24y^3$

63. $y^3 + 2y^2 - 4y - 8$

64. $2x(x - 4)^3 + 3x^2(x - 4)^2$

In Problems 65–69, perform the indicated operations and reduce to lowest terms. Represent all compound fractions as simple fractions reduced to lowest terms.

65. $\dfrac{3x^2(x + 2)^2 - 2x(x + 2)^3}{x^4}$

66. $\dfrac{m - 1}{m^2 - 4m + 4} + \dfrac{m + 3}{m^2 - 4} + \dfrac{2}{2 - m}$

67. $\dfrac{y}{x^2} \div \left(\dfrac{x^2 + 3x}{2x^2 + 5x - 3} \div \dfrac{x^3y - x^2y}{2x^2 - 3x + 1}\right)$

68. $\dfrac{1 - \dfrac{1}{1 + \dfrac{x}{y}}}{1 - \dfrac{1}{1 - \dfrac{x}{y}}}$

69. $\dfrac{a^{-1} - b^{-1}}{ab^{-2} - ba^{-2}}$

70. Check the following solution. If it is wrong, explain what is wrong and how it can be corrected, and then show a correct solution.

$$\frac{x^2 + 2x}{x^2 + x - 2} + x + 2 = \frac{x^2 + 3x + 2}{x^2 + x - 2} = \frac{x + 1}{x - 1}$$

In Problems 71–76, perform the indicated operations, simplify and write answers using positive exponents only. All variables represent positive real numbers.

71. $\left(\dfrac{8u^{-1}}{2^2u^2v^0}\right)^{-2}\left(\dfrac{u^{-5}}{u^{-3}}\right)^3$

72. $\dfrac{5^0}{3^2} + \dfrac{3^{-2}}{2^{-2}}$

73. $\left(\dfrac{27x^2y^{-3}}{8x^{-4}y^3}\right)^{1/3}$

74. $(a^{-1/3}b^{1/4})(9a^{1/3}b^{-1/2})^{3/2}$

75. $(x^{1/2} + y^{1/2})^2$

76. $(3x^{1/2} - y^{1/2})(2x^{1/2} + 3y^{1/2})$

77. Convert to scientific notation and simplify:

$$\frac{0.000\ 000\ 000\ 52}{(1{,}300)(0.000\ 002)}$$

Evaluate Problems 78–85 to four significant digits using a calculator.

78. $\dfrac{(20{,}410)(0.000\ 003\ 477)}{0.000\ 000\ 022\ 09}$

79. 0.1347^5

80. $(-60.39)^{-3}$

81. $82.45^{8/3}$

82. $(0.000\ 000\ 419\ 9)^{2/7}$

83. $\sqrt[5]{0.006\ 604}$

84. $\sqrt[3]{3 + \sqrt{2}}$

85. $\dfrac{2^{-1/2} - 3^{-1/2}}{2^{-1/3} + 3^{-1/3}}$

In Problems 86–94, perform the indicated operations and express answers in simplified form. All radicands represent positive real numbers.

86. $-2x\sqrt[5]{3^6x^7y^{11}}$

87. $\dfrac{2x^2}{\sqrt[3]{4x}}$

88. $\sqrt[5]{\dfrac{3y^2}{8x^2}}$

89. $\sqrt[9]{8x^6y^{12}}$

90. $\sqrt{\sqrt[3]{4x^4}}$

91. $(2\sqrt{x} - 5\sqrt{y})(\sqrt{x} + \sqrt{y})$

92. $\dfrac{3\sqrt{x}}{2\sqrt{x} - \sqrt{y}}$

93. $\dfrac{2\sqrt{u} - 3\sqrt{v}}{2\sqrt{u} + 3\sqrt{v}}$

94. $\dfrac{y^2}{\sqrt{y^2 + 4} - 2}$

95. Rationalize the numerator: $\dfrac{\sqrt{t} - \sqrt{5}}{t - 5}$

96. Write in the form $ax^p + bx^q$, where a and b are real numbers and p and q are rational numbers:

$$\frac{4\sqrt{x} - 3}{2\sqrt{x}}$$

97. Write the repeating decimal 0.545454 . . . in the form a/b reduced to lowest terms, where a and b are positive integers. Is the number rational or irrational?

98. If $M = \{-4, -3, 2\}$ and $N = \{-3, 0, 2\}$, find:
(A) $\{x \mid x \in M \quad \text{or} \quad x \in N\}$
(B) $\{x \mid x \in M \quad \text{and} \quad x \in N\}$

99. Evaluate $x^2 - 4x + 1$ for $x = 2 - \sqrt{3}$.

100. Simplify: $x(2x - 1)(x + 3) - (x - 1)^3$

101. Factor completely with respect to the integers:

$$4x(a^2 - 4a + 4) - 9x^3$$

102. Evaluate each expression on a calculator and determine which pairs have the same value. Verify these results algebraically.
(A) $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$
(B) $\sqrt{4+\sqrt{15}}+\sqrt{4-\sqrt{15}}$
(C) $\sqrt{10}$

In Problems 103–106, simplify and express answers using positive exponents only (m is an integer greater than 1).

103. $\dfrac{8(x-2)^{-3}(x+3)^{2}}{12(x-2)^{-4}(x+3)^{-2}}$ **104.** $\left(\dfrac{a^{-2}}{b^{-1}}+\dfrac{b^{-2}}{a^{-1}}\right)^{-1}$

105. $(x^{1/3}-y^{1/3})(x^{2/3}+x^{1/3}y^{1/3}+y^{2/3})$

106. $\left(\dfrac{x^{m^2}}{x^{2m-1}}\right)^{1/(m-1)} \quad m>1$

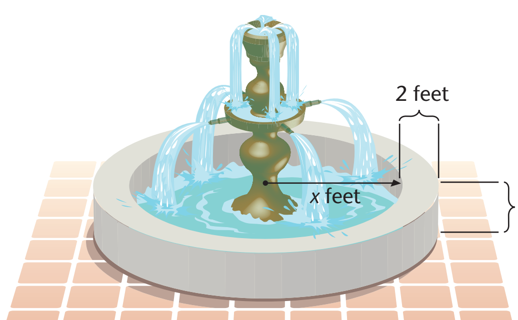
107. Rationalize the denominator: $\dfrac{1}{1-\sqrt[3]{x}}$

108. Rationalize the numerator: $\dfrac{\sqrt[3]{t}-\sqrt[3]{5}}{t-5}$

109. Write in simplified form: $\sqrt[n+1]{x^{n^2}x^{2n+1}} \quad n>0$

APPLICATIONS

110. CONSTRUCTION A circular fountain in a park includes a concrete wall that is 3 feet high and 2 feet thick (see the figure). If the inner radius of the wall is x feet, write an algebraic expression in terms of x that represents the volume of the concrete used to construct the wall. Simplify the expression.



111. ECONOMICS If in the United States in 2003 the total personal income was about \$9,208,000,000,000 and the population was about 291,000,000, estimate to three significant digits the average personal income. Write your answer in scientific notation and in standard decimal form.

112. ECONOMICS The number of units N produced by a petroleum company from the use of x units of capital and y units of labor is approximated by

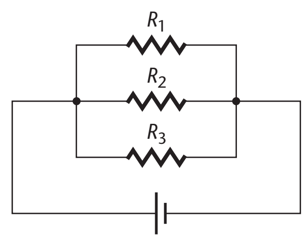
$$N = 20x^{1/2}y^{1/2}$$

(A) Estimate the number of units produced by using 1,600 units of capital and 900 units of labor.
(B) What is the effect on production if the number of units of capital and labor are doubled to 3,200 units and 1,800 units, respectively?
(C) What is the effect on production of doubling the units of labor and capital at any production level?

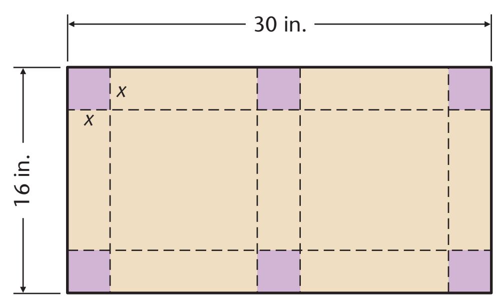
113. ELECTRIC CIRCUIT If three electric resistors with resistances R_1, R_2, and R_3 are connected in parallel, then the total resistance R for the circuit shown in the figure is given by

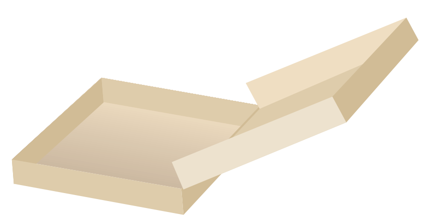
$$R = \frac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}}$$

Represent this compound fraction as a simple fraction.



★**114. CONSTRUCTION** A box with a hinged lid is to be made out of a piece of cardboard that measures 16 by 30 inches. Six squares, x inches on a side, will be cut from each corner and the middle, and then the ends and sides will be folded up to form the box and its lid (see the figure). Express each of the following quantities as a polynomial in both factored and expanded form.
(A) The area of cardboard after the corners have been removed.
(B) The volume of the box.





››› GROUP ACTIVITY Rational and Irrational Numbers

The set of real numbers can be partitioned into two disjoint subsets, the set of rational numbers and the set of irrational numbers. In this activity, we explore the connections between two different methods for representing rational numbers, and we prove that $\sqrt{2}$ is irrational.

Rational numbers can be represented in two ways: as fractions, where both the numerator and denominator are integers, and as decimal expansions that are either repeating or terminating. Consider the rational number $r = a/b$, where a and b are integers with no common factor and $b \neq 0$.

1. If $b = 10^n$ for a positive integer n, what kind of decimal expansion will r have?

2. If $b = 2^m 5^n$ for positive integers m and n, what kind of decimal expansion will r have?

3. If r has a terminating decimal expansion, show that r can be expressed in the form a/b, where $b = 2^m 5^n$.

4. Find a and b for the following repeating expansions (the overbar indicates the repeating block):
(A) $0.\overline{63}$ (B) $0.\overline{486}$ (C) $0.\overline{846153}$

5. Find a and b for $r = 0.1\overline{9}$ and then find a terminating decimal expansion for r.

By the Pythagorean theorem, if the sides of a square have length 1, then the hypotenuse has length $\sqrt{2}$. The number $\sqrt{2}$ is approximately equal to 1.41421, but is not *exactly* equal to 1.41421. Why not? We suppose that $\sqrt{2} = 1.41421$ and show that this assumption leads to a contradiction:

$$\sqrt{2} = \frac{141{,}421}{100{,}000} \quad \text{Square both sides.}$$

$$2 = \frac{141{,}421^2}{100{,}000^2} \quad \text{Multiply both sides by } 100{,}000^2.$$

$$2(100{,}000)^2 = 141{,}421^2 \quad \text{Contradiction!}$$

The last equation is a contradiction (the left-hand side is even but the right-hand side is odd), and therefore $\sqrt{2}$ is not exactly equal to 1.41421.

By similar reasoning we can show that $\sqrt{2}$ is not equal to any rational number. First note that any integer is either even (belongs to $\{0, \pm 2, \pm 4, \ldots\}$) or odd (belongs to $\{\pm 1, \pm 3, \pm 5, \ldots\}$), but not both. And any even integer is either twice an even (belongs to $\{0, \pm 4, \pm 8, \ldots\}$) or twice an odd (belongs to $\{\pm 2, \pm 6, \pm 10\}$), but not both.

6. In the following proof that $\sqrt{2}$ is irrational, explain case 2 and complete cases 3 and 4 by showing that each leads to a contradiction.

Suppose that $\sqrt{2}$ is rational, that is, suppose that $\sqrt{2} = a/b$, where a and b are integers and $b \neq 0$. We will show that this assumption leads to a contradiction.

Case 1. Suppose a and b are both even. Then a/b can be reduced until either the numerator or denominator is odd, and one of the following cases holds.
Case 2. Suppose a is odd and b is even. Then $2b^2 = a^2$. But $2b^2$ is even and a^2 is odd (contradiction!).
Case 3. Suppose a and b are both odd.
Case 4. Suppose a is even and b is odd.