# Trigonometric Identities and Conditional Equations

CHAPTER 7

**TRIGONOMETRIC** functions are widely used in solving real-world problems and in the development of mathematics. Whatever their use, it is often of value to be able to change a trigonometric expression from one form to an equivalent more useful form. This involves the use of identities. Recall that an equation in one or more variables is said to be an **identity** if the left side is equal to the right side for all replacements of the variables for which both sides are defined.

For example, the equation

$$\sin^2 x + \cos^2 x = 1$$

is an identity, but the equation

$$\sin x + \cos x = 1$$

is not. The latter equation is called a **conditional equation,** because it holds for certain values of $x$ (for example, $x = 0$ and $x = \pi/2$) but not for other values for which both sides are defined (for example, $x = \pi/4$). Sections 1 through 4 of Chapter 7 deal with trigonometric identities, and Section 7-5 with conditional trigonometric equations.

## SECTIONS

# 7-1 Basic Identities and Their Use

- Basic Identities
- Establishing Other Identities

In Section 7-1 we review the basic identities introduced in Section 6-4 and show how they are used to establish other identities.

## Basic Identities

In the box we list for convenient reference the basic identities introduced in Section 6-4. These identities will be used very frequently in the work that follows and should be memorized.

### BASIC TRIGONOMETRIC IDENTITIES

**Reciprocal identities**

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

**Quotient identities**

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

**Identities for negatives**

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

**Pythagorean identities**

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

All these identities, with the exception of the second and third Pythagorean identities, were established in Section 6-4. The two exceptions can be derived from the first Pythagorean identity (see Explore-Discuss 1 and Problems 95 and 96 in Exercises 7-1).

### ››› EXPLORE-DISCUSS 1

Discuss an easy way to recall the second and third Pythagorean identities from the first. [*Hint:* Divide through the first Pythagorean identity by appropriate expressions.]

## › Establishing Other Identities

Identities are established to convert one form to an equivalent form that may be more useful. To *verify an identity* means to prove that both sides of an equation are equal for all replacements of the variables for which both sides are defined. Such a proof might use basic identities or other verified identities and algebraic operations such as multiplication, factoring, combining and reducing fractions, and so on. Examples 1 through 6 illustrate some of the techniques used to verify certain identities. The steps illustrated are not necessarily unique—often, there is more than one path to a desired goal. To become proficient in the use of identities, it is important that you work out many problems on your own.

**EXAMPLE 1**

### Identity Verification

Verify the identity $\cos x \tan x = \sin x$.

VERIFICATION

Generally, we proceed by starting with the more complicated of the two sides, and transform that side into the other side in one or more steps using basic identities, algebra, or other established identities. Here we start with the left-hand side and use a quotient identity to rewrite $\tan x$:

$$\cos x \tan x = \cos x \frac{\sin x}{\cos x} \qquad \text{Use algebra.}$$
$$= \sin x$$

**MATCHED PROBLEM 1**

Verify the identity $\sin x \cot x = \cos x$.

**Technology Connections**

Graph the left and right sides of the identity in Example 1 in a graphing calculator by letting $y_1 = \cos x \tan x$ and $y_2 = \sin x$. Use TRACE, moving back and forth between the graphs of $y_1$ and $y_2$, to compare values of $y$ for given values of $x$. What does this investigation illustrate?

## EXAMPLE 2 Identity Verification

Verify the identity $\sec(-x) = \sec x$.

### VERIFICATION

We start with the left-hand side and use a reciprocal identity:

$$\begin{aligned} \sec(-x) &= \frac{1}{\cos(-x)} && \text{Use an identity for negatives.} \\ &= \frac{1}{\cos x} && \text{Use a reciprocal identity.} \\ &= \sec x \end{aligned}$$

## MATCHED PROBLEM 2

Verify the identity $\csc(-x) = -\csc x$.

## EXAMPLE 3 Identity Verification

Verify the identity $\cot x \cos x + \sin x = \csc x$.

### VERIFICATION

We start with the left-hand side and use a quotient identity to rewrite $\cot x$:

$$\begin{aligned} \cot x \cos x + \sin x &= \frac{\cos x}{\sin x} \cos x + \sin x && \text{Use algebra.} \\ &= \frac{\cos^2 x}{\sin x} + \sin x && \text{Write as a single fraction.} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x} && \text{Use } \sin^2 x + \cos^2 x = 1. \\ &= \frac{1}{\sin x} && \text{Use a reciprocal identity.} \\ &= \csc x \end{aligned}$$

### KEY ALGEBRAIC STEPS IN EXAMPLE 3

$$\frac{a}{b} a + b = \frac{a^2}{b} + b = \frac{a^2 + b^2}{b}$$

## MATCHED PROBLEM 3

Verify the identity $\tan x \sin x + \cos x = \sec x$.

To verify an identity, proceed from one side to the other making sure all steps are reversible. Do not use properties of equality to perform the same operation on both sides of the equation. Although there is no fixed method of verification that works for all identities, there are certain steps that help in many cases.

### › SUGGESTED STEPS IN VERIFYING IDENTITIES

1. Start with the more complicated side of the identity, and transform it into the simpler side.
2. Try algebraic operations such as multiplying, factoring, combining fractions, and splitting fractions.
3. If other steps fail, express each function in terms of sine and cosine functions, and then perform appropriate algebraic operations.
4. At each step, keep the other side of the identity in mind. This often reveals what you should do to get there.

## EXAMPLE 4 Identity Verification

Verify the identity $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = 2 \sec x$.

### VERIFICATION

We start with the left-hand side and write it as a single fraction:

$$\begin{aligned}
\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} &= \frac{(1 + \sin x)^2 + \cos^2 x}{\cos x\,(1 + \sin x)} && \text{Expand the numerator.}\\
&= \frac{1 + 2\sin x + \sin^2 x + \cos^2 x}{\cos x\,(1 + \sin x)} && \text{Use } \sin^2 x + \cos^2 x = 1.\\
&= \frac{1 + 2\sin x + 1}{\cos x\,(1 + \sin x)} && \text{Simplify.}\\
&= \frac{2 + 2\sin x}{\cos x\,(1 + \sin x)} && \text{Factor.}\\
&= \frac{2(1 + \sin x)}{\cos x\,(1 + \sin x)} && \text{Simplify.}\\
&= \frac{2}{\cos x} && \text{Use a reciprocal identity.}\\
&= 2 \sec x
\end{aligned}$$

KEY ALGEBRAIC STEPS IN EXAMPLE 4

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ba} \qquad (1 + c)^2 = 1 + 2c + c^2 \qquad \frac{m(a + b)}{n(a + b)} = \frac{m}{n}$$

## MATCHED PROBLEM 4

Verify the identity $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = 2 \csc x$.

## EXAMPLE 5 Identity Verification

Verify the identity $\dfrac{\sin^2 x + 2 \sin x + 1}{\cos^2 x} = \dfrac{1 + \sin x}{1 - \sin x}$.

### VERIFICATION

We start with the left-hand side and factor its numerator:

$$\begin{aligned} \frac{\sin^2 x + 2 \sin x + 1}{\cos^2 x} &= \frac{(\sin x + 1)^2}{\cos^2 x} && \text{Use } \sin^2 x + \cos^2 x = 1. \\ &= \frac{(\sin x + 1)^2}{1 - \sin^2 x} && \text{Factor the denominator.} \\ &= \frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)} && \text{Simplify.} \\ &= \frac{1 + \sin x}{1 - \sin x} \end{aligned}$$

KEY ALGEBRAIC STEPS IN EXAMPLE 5

$$a^2 + 2a + 1 = (a + 1)^2 \qquad 1 - b^2 = (1 - b)(1 + b)$$

## MATCHED PROBLEM 5

Verify the identity $\sec^4 x - 2 \sec^2 x \tan^2 x + \tan^4 x = 1$.

## EXAMPLE 6 Identity Verification

Verify the identity $\dfrac{\tan x - \cot x}{\tan x + \cot x} = 1 - 2 \cos^2 x$.

VERIFICATION

We start with the left-hand side and change to sines and cosines using quotient identities:

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$ Multiply numerator and denominator by (sin $x$)(cos $x$).

$$= \frac{(\sin x)(\cos x)\left(\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}\right)}{(\sin x)(\cos x)\left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right)}$$ Use algebra to transform the compound fraction into a simple fraction.

$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$ Use $\sin^2 x + \cos^2 x = 1$.

$$= \frac{1 - \cos^2 x - \cos^2 x}{1}$$ Simplify.

$$= 1 - 2\cos^2 x$$

KEY ALGEBRAIC STEPS IN EXAMPLE 6

$$\frac{\dfrac{a}{b} - \dfrac{b}{a}}{\dfrac{a}{b} + \dfrac{b}{a}} = \frac{ab\left(\dfrac{a}{b} - \dfrac{b}{a}\right)}{ab\left(\dfrac{a}{b} + \dfrac{b}{a}\right)} = \frac{a^2 - b^2}{a^2 + b^2}$$

## MATCHED PROBLEM 6

Verify the identity $\cot x - \tan x = \dfrac{2\cos^2 x - 1}{\sin x \cos x}$.

**Just observing how others verify identities won't make *you* good at it. You must verify a large number on your own. With practice the process will seem less complicated.**

## EXAMPLE 7 Determining Whether an Equation is an Identity

Determine whether each equation is an identity. If the equation is an identity, verify it. If the equation is not an identity, find a value of $x$ for which both sides are defined but are not equal.

(A) $\tan x + 1 = (\sec x)(\cos x - \sin x)$

(B) $\tan x - 1 = (\sec x)(\sin x - \cos x)$

**SOLUTIONS**

(A) We select several values of $x$ (for example, $x = 0, \pi, \pi/2, \pi/4, \pi/6$) and calculate both the left and right sides of the equation

$$\tan x + 1 = (\sec x)(\cos x - \sin x)$$

Let $x = 0$.

$$\text{Left side:} \quad \tan 0 + 1 = 1$$
$$\text{Right side:} \quad (\sec 0)(\cos 0 - \sin 0) = 1$$

Let $x = \pi$.

$$\text{Left side:} \quad \tan \pi + 1 = 1$$
$$\text{Right side:} \quad (\sec \pi)(\cos \pi - \sin \pi) = 1$$

Let $x = \pi/2$.

$$\text{Left side:} \quad \tan \pi/2 + 1 = \text{Undefined}$$
$$\text{Right side:} \quad (\sec \pi/2)(\cos \pi/2 - \sin \pi/2) = \text{Undefined}$$

Let $x = \pi/4$.

$$\text{Left side:} \quad \tan \pi/4 + 1 = 2$$
$$\text{Right side:} \quad (\sec \pi/4)(\cos \pi/4 - \sin \pi/4) = 0$$

We have found a value of $x$, namely $\pi/4$, for which both sides are defined but are not equal. Therefore the equation is *not* an identity. No further calculation is required.

(B) We select several values of $x$ and calculate both the left and right sides of the equation.

$$\tan x - 1 = (\sec x)(\sin x - \cos x)$$

If $x = 0$ or $x = \pi$, both sides equal $-1$.

If $x = \pi/2$, both sides are undefined.

If $x = \pi/4$, both sides equal 0.

If $x = \pi/6$, both sides equal $\dfrac{1}{\sqrt{3}} - 1$.

These calculations suggest that the equation is an identity, which we now verify. We start with the right-hand side and use a quotient identity to rewrite sec $x$:

$$\begin{aligned}(\sec x)(\sin x - \cos x) &= \left(\frac{1}{\cos x}\right)(\sin x - \cos x) && \text{Distribute.}\\ &= \left(\frac{\sin x}{\cos x} - \frac{\cos x}{\cos x}\right) && \text{Use a quotient identity.}\\ &= \tan x - 1\end{aligned}$$

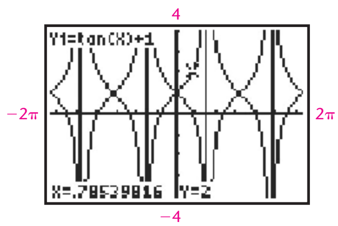
## MATCHED PROBLEM 7

Determine whether each equation is an identity. If the equation is an identity, verify it. If the equation is not an identity, find a value of $x$ for which both sides are defined but are not equal.

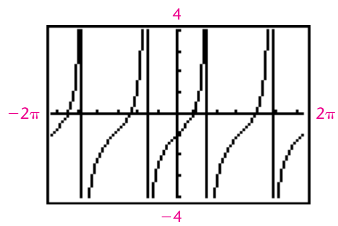
(A) $\dfrac{\sin x}{1 - \cos^2 x} = \csc x$ (B) $\dfrac{\sin x}{1 - \cos^2 x} = \sec x$

### Technology Connections

Using a graphing calculator, we can eliminate the calculations of Example 7 by comparing the graphs of each side of the given equation. Figure 1 shows the graphs of each side of the equation of Example 7(A). The equation is not an identity because the graphs do not coincide (note that when $x = \pi/4$, $Y_1$ has the value 2 but $Y_2$ has the value 0). Figure 2 shows the graphs of each side of the equation of Example 7(B). The equation appears to be an identity because the graphs coincide; to show that it is indeed an identity, it must still be verified as in the solution to Example 7(B).




› Figure 1




› Figure 2

## ANSWERS TO MATCHED PROBLEMS

*In the following identity verifications, other correct sequences of steps are possible—the process is not unique.*

1. $\sin x \cot x = \sin x \dfrac{\cos x}{\sin x} = \cos x$

2. $\csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\csc x$

3. $\tan x \sin x + \cos x = \dfrac{\sin^2 x}{\cos x} + \cos x = \dfrac{\sin^2 x + \cos^2 x}{\cos x} = \dfrac{1}{\cos x} = \sec x$

**4.** $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = \dfrac{(1 + \cos x)^2 + \sin^2 x}{\sin x\,(1 + \cos x)} = \dfrac{1 + 2\cos x + \cos^2 x + \sin^2 x}{\sin x\,(1 + \cos x)}$

$= \dfrac{2\,(1 + \cos x)}{\sin x\,(1 + \cos x)} = 2 \csc x$

**5.** $\sec^4 x - 2\sec^2 x \tan^2 x + \tan^4 x = (\sec^2 x - \tan^2 x)^2 = 1^2 = 1$

**6.** $\cot x - \tan x = \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x} = \dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

$= \dfrac{\cos^2 x - (1 - \cos^2 x)}{\sin x \cos x} = \dfrac{2\cos^2 x - 1}{\sin x \cos x}$

**7.** (A) An identity: $\dfrac{\sin x}{1 - \cos^2 x} = \dfrac{\sin x}{\sin^2 x} = \dfrac{1}{\sin x} = \csc x$

(B) Not an identity: the left side is not equal to the right side for $x = \pi/6$, for example.

# 7-1 Exercises

*Verify that Problems 1–26 are identities.*

**1.** $\sin\theta \sec\theta = \tan\theta$

**2.** $\cos\theta \csc\theta = \cot\theta$

**3.** $\cot u \sec u \sin u = 1$

**4.** $\tan\theta \csc\theta \cos\theta = 1$

**5.** $\dfrac{\sin(-x)}{\cos(-x)} = -\tan x$

**6.** $\cot(-x)\tan x = -1$

**7.** $\sin\alpha = \dfrac{\tan\alpha \cot\alpha}{\csc\alpha}$

**8.** $\tan\alpha = \dfrac{\cos\alpha \sec\alpha}{\cot\alpha}$

**9.** $\cot u + 1 = (\csc u)(\cos u + \sin u)$

**10.** $\tan u + 1 = (\sec u)(\sin u + \cos u)$

**11.** $\dfrac{\cos x - \sin x}{\sin x \cos x} = \csc x - \sec x$

**12.** $\dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \cot x - \tan x$

**13.** $\dfrac{\sin^2 t}{\cos t} + \cos t = \sec t$

**14.** $\dfrac{\cos^2 t}{\sin t} + \sin t = \csc t$

**15.** $\dfrac{\cos x}{1 - \sin^2 x} = \sec x$

**16.** $\dfrac{\sin u}{1 - \cos^2 u} = \csc u$

**17.** $(1 - \cos u)(1 + \cos u) = \sin^2 u$

**18.** $(1 - \sin t)(1 + \sin t) = \cos^2 t$

**19.** $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

**20.** $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

**21.** $(\sec t + 1)(\sec t - 1) = \tan^2 t$

**22.** $(\csc t - 1)(\csc t + 1) = \cot^2 t$

**23.** $\csc^2 x - \cot^2 x = 1$

**24.** $\sec^2 u - \tan^2 u = 1$

**25.** $\cot x + \sec x = \dfrac{\cos x + \tan x}{\sin x}$

**26.** $\sin m\,(\csc m - \sin m) = \cos^2 m$

*In Problems 27–34, show that the equation is not an identity by finding a value of x for which both sides are defined but are not equal.*

**27.** $\sqrt{x^2} = x$

**28.** $\sqrt{x^2 + 4x + 4} = x + 2$

**29.** $x^4 + x^3 - x^2 - x = x^5 + x^4 - x^3 - x^2$

**30.** $x^5 - 3x^2 + 2x = x^6 - 3x^4 + 2x^2$

**31.** $\sin x \cos x = \tan x$

**32.** $\cot x \csc x = \sec x$

**33.** $\cos x = 1 - \sin x$

**34.** $1 + \sin x = \tan x + \sec x$

*In Problems 35–38, graph all parts of each problem in the same viewing window in a graphing calculator.*

**35.** $-\pi \le x \le \pi$
(A) $y = \sin^2 x$ (B) $y = \cos^2 x$
(C) $y = \sin^2 x + \cos^2 x$

**36.** $-\pi \le x \le \pi$
(A) $y = \sec^2 x$ (B) $y = \tan^2 x$
(C) $y = \sec^2 x - \tan^2 x$

**37.** $-\pi \le x \le \pi$
(A) $y = \dfrac{\cos x}{\cot x \sin x}$ (B) $y = 1$

**38.** $-\pi \le x \le \pi$
(A) $y = \dfrac{\sin x}{\cos x \tan x}$ (B) $y = 1$

*In Problems 39–46, is the equation an identity? Explain.*

**39.** $\frac{x^2 - 9}{x + 3} = x - 3$

**40.** $\frac{5x}{|x|} = 5$

**41.** $\sqrt{x^2 + 4x + 4} = x + 2$

**42.** $\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{|x|}$

**43.** $\sin x - \cos x = 1$

**44.** $\sin x + \cos x = 1$

**45.** $\sin^2 x - \cos^2 x = 1$

**46.** $\sin^3 x + \cos^3 x = 1$

*Verify that Problems 47–76 are identities.*

**47.** $\frac{1 - (\sin x - \cos x)^2}{\sin x} = 2 \cos x$

**48.** $\frac{1 - \cos^2 y}{(1 - \sin y)(1 + \sin y)} = \tan^2 y$

**49.** $\cos \theta + \sin \theta = \frac{\cot \theta + 1}{\csc \theta}$

**50.** $\sin \theta + \cos \theta = \frac{\tan \theta + 1}{\sec \theta}$

**51.** $\frac{1 + \cos y}{1 - \cos y} = \frac{\sin^2 y}{(1 - \cos y)^2}$

**52.** $1 - \sin y = \frac{\cos^2 y}{1 + \sin y}$

**53.** $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

**54.** $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$

**55.** $\frac{\csc \theta}{\cot \theta + \tan \theta} = \cos \theta$

**56.** $\frac{1 + \sec \theta}{\sin \theta + \tan \theta} = \csc \theta$

**57.** $\ln (\tan x) = \ln (\sin x) - \ln (\cos x)$

**58.** $\ln (\cot x) = \ln (\cos x) - \ln (\sin x)$

**59.** $\ln (\cot x) = -\ln (\tan x)$

**60.** $\ln (\csc x) = -\ln (\sin x)$

**61.** $\frac{1 - \cos A}{1 + \cos A} = \frac{\sec A - 1}{\sec A + 1}$

**62.** $\frac{1 - \csc y}{1 + \csc y} = \frac{\sin y - 1}{\sin y + 1}$

**63.** $\sin^4 w - \cos^4 w = 1 - 2 \cos^2 w$

**64.** $\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x = 1$

**65.** $\sec x - \frac{\cos x}{1 + \sin x} = \tan x$

**66.** $\csc n - \frac{\sin n}{1 + \cos n} = \cot n$

**67.** $\frac{\cos^2 z - 3 \cos z + 2}{\sin^2 z} = \frac{2 - \cos z}{1 + \cos z}$

**68.** $\frac{\sin^2 t + 4 \sin t + 3}{\cos^2 t} = \frac{3 + \sin t}{1 - \sin t}$

**69.** $\frac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} = 1 + \sin \theta \cos \theta$

**70.** $\frac{\cos^3 u + \sin^3 u}{\cos u + \sin u} = 1 - \sin u \cos u$

**71.** $(\sec x - \tan x)^2 = \frac{1 - \sin x}{1 + \sin x}$

**72.** $(\cot u - \csc u)^2 = \frac{1 - \cos u}{1 + \cos u}$

**73.** $\frac{\csc^4 x - 1}{\cot^2 x} = 2 + \cot^2 x$

**74.** $\frac{\sec^4 x - 1}{\tan^2 x} = 2 + \tan^2 x$

**75.** $\frac{1 + \sin v}{\cos v} = \frac{\cos v}{1 - \sin v}$

**76.** $\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$

*In Problems 77–88, use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.*

**77.** $\frac{\sin (-x)}{\cos (-x) \tan (-x)} = -1$

**78.** $\frac{\cos (-x)}{\sin x \cot (-x)} = 1$

**79.** $\frac{\sin x}{\cos x \tan (-x)} = -1$

**80.** $\frac{\cos x}{\sin (-x) \cot (-x)} = 1$

**81.** $\sin x + \frac{\cos^2 x}{\sin x} = \sec x$

**82.** $\frac{1 - \tan^2 x}{1 - \cot^2 x} = \tan^2 x$

**83.** $\sin x + \frac{\cos^2 x}{\sin x} = \csc x$

**84.** $\frac{\tan^2 x - 1}{1 - \cot^2 x} = \tan^2 x$

**85.** $\frac{\tan x}{\sin x - 2 \tan x} = \frac{1}{\cos x - 2}$

**86.** $\frac{\cos x}{1 - \sin x} + \frac{\cos x}{1 + \sin x} = 2 \sec x$

**87.** $\dfrac{\tan x}{\sin x + 2\tan x} = \dfrac{1}{\cos x - 2}$

**88.** $\dfrac{\cos x}{\sin x + 1} - \dfrac{\cos x}{\sin x - 1} = 2\csc x$

*Verify that Problems 89–94 are identities.*

**89.** $\dfrac{2\sin^2 x + 3\cos x - 3}{\sin^2 x} = \dfrac{2\cos x - 1}{1 + \cos x}$

**90.** $\dfrac{3\cos^2 z + 5\sin z - 5}{\cos^2 z} = \dfrac{3\sin z - 2}{1 + \sin z}$

**91.** $\dfrac{\tan u + \sin u}{\tan u - \sin u} - \dfrac{\sec u + 1}{\sec u - 1} = 0$

**92.** $\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

**93.** $\tan\alpha + \cot\beta = \dfrac{\tan\beta + \cot\alpha}{\tan\beta\cot\alpha}$

**94.** $\dfrac{\cot\alpha + \cot\beta}{\cot\alpha\cot\beta - 1} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

*In Problems 95 and 96, fill in the blanks citing the appropriate basic trigonometric identity.*

**95.**

| Statement | Reason |
|---|---|
| $\cot^2 x + 1 = \left(\dfrac{\cos x}{\sin x}\right)^2 + 1$ | (A) ______ |
| $= \dfrac{\cos^2 x}{\sin^2 x} + 1$ | Algebra |
| $= \dfrac{\cos^2 x + \sin^2 x}{\sin^2 x}$ | Algebra |
| $= \dfrac{1}{\sin^2 x}$ | (B) ______ |
| $= \left(\dfrac{1}{\sin x}\right)^2$ | Algebra |
| $= \csc^2 x$ | (C) ______ |

**96.**

| Statement | Reason |
|---|---|
| $\tan^2 x + 1 = \left(\dfrac{\sin x}{\cos x}\right)^2 + 1$ | (A) ______ |
| $= \dfrac{\sin^2 x}{\cos^2 x} + 1$ | Algebra |
| $= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}$ | Algebra |
| $= \dfrac{1}{\cos^2 x}$ | (B) ______ |
| $= \left(\dfrac{1}{\cos x}\right)^2$ | Algebra |
| $= \sec^2 x$ | (C) ______ |

*In Problems 97–102, examine the graph of $f(x)$ in a graphing calculator to find a function of the form $g(x) = k + AT(x)$ that has the same graph as $f(x)$, where $k$ and $A$ are constants and $T(x)$ is one of the six trigonometric functions. Verify the identity $f(x) = g(x)$.*

**97.** $f(x) = \dfrac{1 - \sin^2 x}{\tan x} + \sin x \cos x$

**98.** $f(x) = \dfrac{1 + \sin x}{2\cos x} - \dfrac{\cos x}{2 + 2\sin x}$

**99.** $f(x) = \dfrac{\cos^2 x}{1 + \sin x - \cos^2 x}$

**100.** $f(x) = \dfrac{\tan x \sin x}{1 - \cos x}$

**101.** $f(x) = \dfrac{1 + \cos x - 2\cos^2 x}{1 - \cos x} - \dfrac{\sin^2 x}{1 + \cos x}$

**102.** $f(x) = \dfrac{3\sin x - 2\sin x\cos x}{1 - \cos x} - \dfrac{1 + \cos x}{\sin x}$

*Each of the equations in Problems 103–110 is an identity in certain quadrants associated with $x$. Indicate which quadrants.*

**103.** $\sqrt{1 - \cos^2 x} = -\sin x$

**104.** $\sqrt{1 - \sin^2 x} = \cos x$

**105.** $\sqrt{1 - \cos^2 x} = \sin x$

**106.** $\sqrt{1 - \sin^2 x} = -\cos x$

**107.** $\sqrt{1 - \sin^2 x} = |\cos x|$

**108.** $\sqrt{1 - \cos^2 x} = |\sin x|$

**109.** $\dfrac{\sin x}{\sqrt{1 - \sin^2 x}} = \tan x$

**110.** $\dfrac{\sin x}{\sqrt{1 - \sin^2 x}} = -\tan x$

*In calculus, trigonometric substitutions provide an effective way to rationalize the radical forms $\sqrt{a^2 - u^2}$ and $\sqrt{a^2 + u^2}$, which in turn leads to the solution to an important class of problems. Problems 111–114 involve such transformations. [Recall: $\sqrt{x^2} = |x|$ for all real numbers $x$.]*

**111.** In the radical form $\sqrt{a^2 - u^2}$, $a > 0$, let $u = a\sin x$, $-\pi/2 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals.

**112.** In the radical form $\sqrt{a^2 - u^2}$, $a > 0$, let $u = a\cos x$, $0 < x < \pi$. Simplify, using a basic identity, and write the final form free of radicals.

**113.** In the radical form $\sqrt{a^2 + u^2}$, $a > 0$, let $u = a\tan x$, $0 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals.

**114.** In the radical form $\sqrt{a^2 + u^2}$, $a > 0$, let $u = a\cot x$, $0 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals.

# 7-2 Sum, Difference, and Cofunction Identities



The basic identities discussed in Section 7-1 involved only one variable. In this section, we consider identities that involve two variables.

## Sum and Difference Identities for Cosine

We start with the important **difference identity for cosine:**

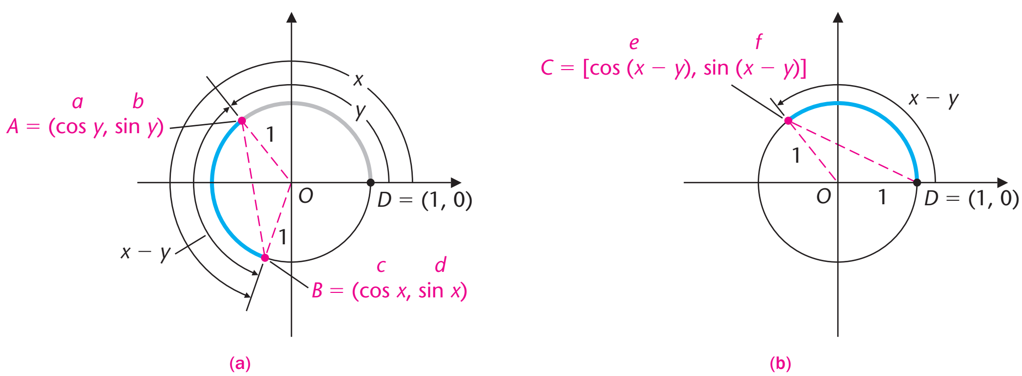
$$\cos(x - y) = \cos x \cos y + \sin x \sin y \tag{1}$$

Many other useful identities can be readily verified from this particular one.

Here, we sketch a proof of equation (1) assuming $x$ and $y$ are in the interval $(0, 2\pi)$ and $x > y > 0$. It then follows easily, by periodicity and basic identities, that equation (1) holds for all real numbers $x$ and $y$.

First, associate $x$ and $y$ with arcs and angles on the unit circle as indicated in Figure 1(a). Using the definitions of the trigonometric functions given in Section 6-2, label the terminal points of $x$ and $y$ as shown in Figure 1(a). To simplify notation we let $a = \cos y$, $b = \sin y$, and so on, as indicated.

**Figure 1** Difference identity.



Now if you rotate the triangle *AOB* clockwise about the origin until the terminal point *A* coincides with $D = (1, 0)$, then terminal point *B* will be at *C*, as shown in Figure 1(b). Thus, because rotation preserves lengths,

$$\begin{aligned} d(A, B) &= d(C, D) \\ \sqrt{(c-a)^2 + (d-b)^2} &= \sqrt{(1-e)^2 + (0-f)^2} \\ (c-a)^2 + (d-b)^2 &= (1-e)^2 + f^2 \\ c^2 - 2ac + a^2 + d^2 - 2db + b^2 &= 1 - 2e + e^2 + f^2 \\ (c^2 + d^2) + (a^2 + b^2) - 2ac - 2db &= 1 - 2e + (e^2 + f^2) \end{aligned} \tag{2}$$

Because points *A*, *B*, and *C* are on unit circles, $c^2 + d^2 = 1$, $a^2 + b^2 = 1$, and $e^2 + f^2 = 1$, and equation (2) simplifies to

$$e = ac + bd \tag{3}$$

Replacing $e$, $a$, $c$, $b$, and $d$ with $\cos(x - y)$, $\cos y$, $\cos x$, $\sin y$, and $\sin x$, respectively (see Fig. 1), we obtain

$$\begin{aligned} \cos(x - y) &= \cos y \cos x + \sin y \sin x \\ &= \cos x \cos y + \sin x \sin y \end{aligned} \tag{4}$$

We have thus established the difference identity for cosine.

If we replace $y$ with $-y$ in equation (4) and use the identities for negatives (a good exercise for you), we obtain

$$\mathbf{\cos(x + y) = \cos x \cos y - \sin x \sin y} \tag{5}$$

This is the **sum identity for cosine.**

### ››› EXPLORE-DISCUSS 1

Discuss how you would show that the equation

$$\cos(x - y) = \cos x - \cos y$$

is *not* an identity.

## › Cofunction Identities

To obtain sum and difference identities for the sine and tangent functions, we first derive *cofunction identities* directly from equation (1), the difference identity for cosine:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \qquad \text{Substitute } x = \frac{\pi}{2}.$$

$$\cos\left(\frac{\pi}{2} - y\right) = \cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y \qquad \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1.$$

$$= (0)(\cos y) + (1)(\sin y) \qquad \text{Simplify.}$$

$$= \sin y$$

Thus, we have the **cofunction identity for cosine:**

$$\cos\left(\frac{\pi}{2} - y\right) = \sin y \tag{6}$$

for $y$ any real number or angle in radian measure. If $y$ is in degree measure, replace $\pi/2$ with 90°.

Now, if we let $y = \pi/2 - x$ in equation (6), we have

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \sin\left(\frac{\pi}{2} - x\right) \qquad \text{Simplify left-hand side.}$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

This is the **cofunction identity for sine;** that is,

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \tag{7}$$

where $x$ is any real number or angle in radian measure. If $x$ is in degree measure, replace $\pi/2$ with 90°.

Finally, we state the **cofunction identity for tangent** (and leave its derivation to Problem 10 in Exercises 7-2):

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \tag{8}$$

for $x$ any real number or angle in radian measure. If $x$ is in degree measure, replace $\pi/2$ with 90°.

## › Sum and Difference Identities for Sine and Tangent

To derive a difference identity for sine, we first substitute $x - y$ for $y$ in equation (6):

$$\sin(x - y) = \cos\left[\frac{\pi}{2} - (x - y)\right] \qquad \text{Use algebra.}$$

$$= \cos\left[\left(\frac{\pi}{2} - x\right) - (-y)\right] \qquad \text{Use equation (1).}$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos(-y) + \sin\left(\frac{\pi}{2} - x\right)\sin(-y) \qquad \text{Use equations (6) and (7) and identities for negatives.}$$

$$= \sin x \cos y - \cos x \sin y$$

The same result is obtained by replacing $\pi/2$ with 90°. Thus,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{9}$$

is the **difference identity for sine.**

Now, if we replace $y$ in equation (9) with $-y$ (a good exercise for you), we obtain

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{10}$$

the **sum identity for sine.**

It is not difficult to derive sum and difference identities for the tangent function. See if you can supply the reason for each step:

$$\begin{aligned}
\tan(x - y) &= \frac{\sin(x - y)}{\cos(x - y)} \\
&= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} \\
&= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}} \\
&= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin x}{\cos y}}{1 + \dfrac{\sin x \sin y}{\cos x \cos y}} \\
&= \frac{\tan x - \tan y}{1 + \tan x \tan y}
\end{aligned}$$

Divide the numerator and denominator by $\cos x$ and $\cos y$.

Thus,

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \tag{11}$$

for all angles or real numbers $x$ and $y$ for which both sides are defined. This is the **difference identity for tangent.**

If we replace $y$ in equation (11) with $-y$ (another good exercise for you), we obtain

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \tag{12}$$

the **sum identity for tangent.**

### ››› EXPLORE-DISCUSS 2

Discuss how you would show that the equation

$$\tan(x - y) = \tan x - \tan y$$

is *not* an identity. How many solutions does the equation have? Explain.

## Summary and Use

Before proceeding with examples illustrating the use of these new identities, review the list given in the box.

### SUMMARY OF IDENTITIES

**Sum identities**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

**Difference identities**

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

**Cofunction identities**

(Replace $\pi/2$ with 90° if $x$ is in degrees.)

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

**EXAMPLE 1** 

### Using the Difference Identity

Simplify $\cos(x - \pi)$ using the difference identity.

**SOLUTION**

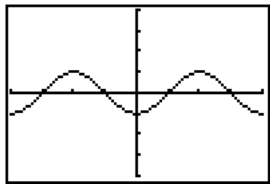
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\cos(x - \pi) = \cos x \cos \pi + \sin x \sin \pi \qquad \cos \pi = -1, \sin \pi = 0$$
$$= (\cos x)(-1) + (\sin x)(0) \qquad \text{Simplify.}$$
$$= -\cos x$$

**MATCHED PROBLEM 1**

Simplify $\sin(x + 3\pi/2)$ using a sum identity.

**Technology Connections**

To check the simplification of Example 1, compare the graphs of $y_1 = \cos(x - \pi)$ and $y_2 = -\cos x$ by using TRACE to move back and forth between $y_1$ and $y_2$ for various values of $x$. Note that the two graphs coincide (Fig. 2).




› **Figure 2**

## EXAMPLE 2 Finding Exact Values

Find the exact value of tan 75° in radical form.

**SOLUTION**

Because we can write $75° = 45° + 30°$, the sum of two special angles, we can use the sum identity for tangents with $x = 45°$ and $y = 30°$:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \text{Substitute } x = 45°,\ y = 30°.$$

$$\tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} \quad \text{Evaluate functions exactly.}$$

$$= \frac{1 + (1/\sqrt{3})}{1 - 1(1/\sqrt{3})} \quad \text{Multiply numerator and denominator by } \sqrt{3} \text{ and simplify.}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \quad \text{Rationalize denominator and simplify.}$$

$$= 2 + \sqrt{3}$$

## MATCHED PROBLEM 2

Find the exact value of cos 15° in radical form.

## EXAMPLE 3 Finding Exact Values

Find the exact value of $\cos(x + y)$, given $\sin x = \frac{3}{5}$, $\cos y = \frac{4}{5}$, $x$ is an angle in quadrant II, and $y$ is an angle in quadrant I. Do not use a calculator.

**SOLUTION**

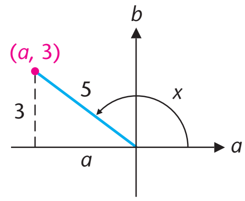
We start with the sum identity for cosine,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

We know sin $x$ and cos $y$ but not cos $x$ and sin $y$. We find the latter two using two different methods as follows (use the method that is easiest for you).

Given $\sin x = \frac{3}{5}$ and $x$ is an angle in quadrant II, find cos $x$:

**METHOD I. USE A REFERENCE TRIANGLE:**




$$\cos x = \frac{a}{5}$$

$$a^2 + 3^2 = 5^2$$

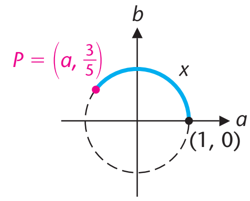
$$a^2 = 16$$

$$a = \pm 4$$

In quadrant II, $a = -4$

Therefore, $\cos x = -\frac{4}{5}$

**METHOD II. USE A UNIT CIRCLE:**




$$\cos x = a$$

$$a^2 + \left(\tfrac{3}{5}\right)^2 = 1$$

$$a^2 = \tfrac{16}{25}$$

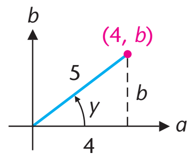
$$a = \pm\tfrac{4}{5}$$

In quadrant II, $a = -\frac{4}{5}$

Therefore, $\cos x = -\frac{4}{5}$

Given $\cos y = \frac{4}{5}$ and $y$ is an angle in quadrant I, find sin $y$:

**METHOD I. USE A REFERENCE TRIANGLE:**




$$\sin y = \frac{b}{5}$$

$$4^2 + b^2 = 5^2$$

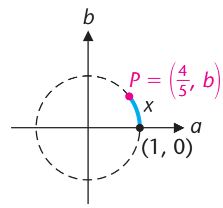
$$b^2 = 9$$

$$b = \pm 3$$

In quadrant I, $b = 3$

Therefore, $\sin y = \frac{3}{5}$

**METHOD II. USE A UNIT CIRCLE:**




$$\sin y = b$$

$$\left(\tfrac{4}{5}\right)^2 + b^2 = 1$$

$$b^2 = \tfrac{9}{25}$$

$$b = \pm\tfrac{3}{5}$$

In quadrant I, $b = \frac{3}{5}$

Therefore, $\sin y = \frac{3}{5}$

We can now evaluate cos $(x + y)$ without knowing $x$ and $y$:

$$\begin{aligned}\cos(x + y) &= \cos x \cos y - \sin x \sin y \\ &= \left(-\tfrac{4}{5}\right)\left(\tfrac{4}{5}\right) - \left(\tfrac{3}{5}\right)\left(\tfrac{3}{5}\right) = -\tfrac{25}{25} = -1\end{aligned}$$

## MATCHED PROBLEM 3

Find the exact value of $\sin(x - y)$, given $\sin x = -\frac{2}{3}$, $\cos y = \sqrt{5}/3$, $x$ is an angle in quadrant III, and $y$ is an angle in quadrant IV. Do not use a calculator.

## EXAMPLE 4 Identity Verification

Verify the identity $\tan x + \cot y = \dfrac{\cos(x - y)}{\cos x \sin y}$.

### VERIFICATION

We start with the right-hand side and use the difference identity for cosine:

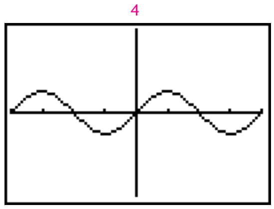
$$
\begin{aligned}
\frac{\cos(x-y)}{\cos x \sin y} &= \frac{\cos x \cos y + \sin x \sin y}{\cos x \sin y} && \text{Write as sum of two fractions.}\\
&= \frac{\cancel{\cos x}\cos y}{\cancel{\cos x}\sin y} + \frac{\sin x \cancel{\sin y}}{\cos x \cancel{\sin y}} && \text{Simplify and use quotient identities.}\\
&= \cot y + \tan x && \text{Commute terms.}\\
&= \tan x + \cot y
\end{aligned}
$$

## MATCHED PROBLEM 4

Verify the identity $\cot y - \cot x = \dfrac{\sin(x - y)}{\sin x \sin y}$.

## ANSWERS TO MATCHED PROBLEMS

**1.** $-\cos x$ **2.** $(1 + \sqrt{3})/2\sqrt{2}$ or $(\sqrt{6} + \sqrt{2})/4$ **3.** $-4\sqrt{5}/9$




**4.** $\dfrac{\sin(x-y)}{\sin x \sin y} = \dfrac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} = \dfrac{\cancel{\sin x}\cos y}{\cancel{\sin x}\sin y} - \dfrac{\cos x \cancel{\sin y}}{\sin x \cancel{\sin y}} = \cot y - \cot x$

# 7-2 Exercises

*In Problems 1–8, show that the equation is not an identity by finding a value of x and a value of y for which both sides are defined but are not equal.*

**1.** $(x + y)^2 = x^2 + y^2$

**2.** $(x - y)^3 = x^3 - y^3$

**3.** $x \sin y = \sin xy$

**4.** $x \tan y = \tan xy$

**5.** $\cos (x + y) = \cos x + \cos y$

**6.** $\tan (x + y) = \tan x + \tan y$

**7.** $\tan (x - y) = \tan x - \tan y$

**8.** $\sin (x - y) = \sin x - \sin y$

*In Problems 9–16, is the equation an identity? Explain, making use of the sum or difference identities.*

**9.** $\tan (x - \pi) = \tan x$

**10.** $\cos (x + \pi) = \cos x$

**11.** $\sin (x - \pi) = \sin x$

**12.** $\cot (x + \pi) = \cot x$

**13.** $\csc (2\pi - x) = \csc x$

**14.** $\sec (2\pi - x) = \sec x$

**15.** $\sin (x - \pi/2) = -\cos x$

**16.** $\cos (x - \pi/2) = -\sin x$

*Verify each identity in Problems 17–20 using cofunction identities for sine and cosine and basic identities discussed in Section 7-1.*

**17.** $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

**18.** $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

**19.** $\csc\left(\frac{\pi}{2} - x\right) = \sec x$

**20.** $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

*Convert Problems 21–26 to forms involving sin x, cos x, and/or tan x using sum or difference identities.*

**21.** $\sin (30° - x)$

**22.** $\sin (x - 45°)$

**23.** $\sin (180° - x)$

**24.** $\cos (x + 180°)$

**25.** $\tan\left(x + \frac{\pi}{3}\right)$

**26.** $\tan\left(\frac{\pi}{4} - x\right)$

*Use appropriate identities to find exact values for Problems 27–34. Do not use a calculator.*

**27.** $\sec 75°$

**28.** $\sin 75°$

**29.** $\sin \frac{7\pi}{12}$ $\left[\textit{Hint}: \frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}\right]$

**30.** $\cos \frac{\pi}{12}$ $\left[\textit{Hint}: \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}\right]$

**31.** $\cos 74° \cos 44° + \sin 74° \sin 44°$

**32.** $\sin 22° \cos 38° + \cos 22° \sin 38°$

**33.** $\dfrac{\tan 27° + \tan 18°}{1 - \tan 27° \tan 18°}$

**34.** $\dfrac{\tan 110° - \tan 50°}{1 + \tan 110° \tan 50°}$

*Find sin (x − y) and tan (x + y) exactly without a calculator using the information given in Problems 35–38.*

**35.** $\sin x = -\frac{3}{5}$, $\sin y = \sqrt{8}/3$, $x$ is a quadrant IV angle, $y$ is a quadrant I angle.

**36.** $\sin x = \frac{2}{3}$, $\cos y = -\frac{1}{4}$, $x$ is a quadrant II angle, $y$ is a quadrant III angle.

**37.** $\tan x = \frac{3}{4}$, $\tan y = -\frac{1}{2}$, $x$ is a quadrant III angle, $y$ is a quadrant IV angle.

**38.** $\cos x = -\frac{1}{3}$, $\tan y = \frac{1}{2}$, $x$ is a quadrant II angle, $y$ is a quadrant III angle.

*Verify each identity in Problems 39–52.*

**39.** $\cos 2x = \cos^2 x - \sin^2 x$

**40.** $\sin 2x = 2 \sin x \cos x$

**41.** $\cot (x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

**42.** $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

**43.** $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

**44.** $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

**45.** $\dfrac{\sin (v + u)}{\sin (v - u)} = \dfrac{\cot u + \cot v}{\cot u - \cot v}$

**46.** $\dfrac{\sin (u + v)}{\sin (u - v)} = \dfrac{\tan u + \tan v}{\tan u - \tan v}$

**47.** $\cot x - \tan y = \dfrac{\cos (x + y)}{\sin x \cos y}$

**48.** $\tan x - \tan y = \dfrac{\sin (x - y)}{\cos x \cos y}$

**49.** $\tan (x - y) = \dfrac{\cot y - \cot x}{\cot x \cot y + 1}$

**50.** $\tan (x + y) = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$

**51.** $\dfrac{\cos (x + h) - \cos x}{h} = \cos x \left(\dfrac{\cos h - 1}{h}\right) - \sin x \left(\dfrac{\sin h}{h}\right)$

**52.** $\dfrac{\sin (x + h) - \sin x}{h} = \sin x \left(\dfrac{\cos h - 1}{h}\right) + \cos x \left(\dfrac{\sin h}{h}\right)$

*Evaluate both sides of the difference identity for sine and the sum identity for tangent for the values of x and y indicated in Problems 53–56. Evaluate to four significant digits using a calculator.*

**53.** $x = 5.288$, $y = 1.769$

**54.** $x = 3.042$, $y = 2.384$

**55.** $x = 42.08°$, $y = 68.37°$

**56.** $x = 128.3°$, $y = 25.62°$

**57.** Show that $\sec (x - y) = \sec x - \sec y$ is *not* an identity.

**58.** Show that $\csc (x + y) = \csc x + \csc y$ is *not* an identity.

*In Problems 59–64, use sum or difference identities to convert each equation to a form involving sin x, cos x, and/or tan x. Enter the original equation in a graphing calculator as $y_1$ and the converted form as $y_2$, then graph $y_1$ and $y_2$ in the same viewing window. Use TRACE to compare the two graphs.*

**59.** $y = \sin (x + \pi/6)$

**60.** $y = \sin (x - \pi/3)$

**61.** $y = \cos (x - 3\pi/4)$

**62.** $y = \cos (x + 5\pi/6)$

**63.** $y = \tan (x + 2\pi/3)$

**64.** $y = \tan (x - \pi/4)$

*In Problems 65–68, evaluate exactly as real numbers without the use of a calculator.*

**65.** $\sin [\cos^{-1} (-\frac{4}{5}) + \sin^{-1} (-\frac{3}{5})]$

**66.** $\cos [\sin^{-1} (-\frac{3}{5}) + \cos^{-1} (\frac{4}{5})]$

**67.** $\sin [\arccos \frac{1}{2} + \arcsin (-1)]$

**68.** $\cos [\arccos (-\sqrt{3}/2) - \arcsin (-\frac{1}{2})]$

**69.** Express $\sin (\sin^{-1} x + \cos^{-1} y)$ in an equivalent form free of trigonometric and inverse trigonometric functions.

**70.** Express $\cos (\sin^{-1} x - \cos^{-1} y)$ in an equivalent form free of trigonometric and inverse trigonometric functions.

*Verify the identities in Problems 71 and 72.*

**71.** $\cos (x + y + z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$

**72.** $\sin (x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$

*In Problems 73 and 74, write each equation in terms of a single trigonometric function. Enter the original equation in a graphing calculator as $y_1$ and the converted form as $y_2$, then graph $y_1$ and $y_2$ in the same viewing window. Use TRACE to compare the two graphs.*

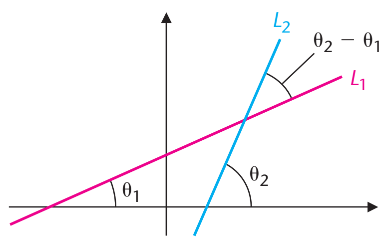
**73.** $y = \cos 1.2x \cos 0.8x - \sin 1.2x \sin 0.8x$

**74.** $y = \sin 0.8x \cos 0.3x - \cos 0.8x \sin 0.3x$

## APPLICATIONS

**75.** **ANALYTIC GEOMETRY** Use the information in the figure to show that

$$\tan (\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_1 m_2}$$




$\tan \theta_1 = \text{Slope of } L_1 = m_1$

$\tan \theta_2 = \text{Slope of } L_2 = m_2$

**76.** ANALYTIC GEOMETRY Find the acute angle of intersection between the two lines $y = 3x + 1$ and $y = \frac{1}{2}x - 1$. (Use the results of Problem 75.)

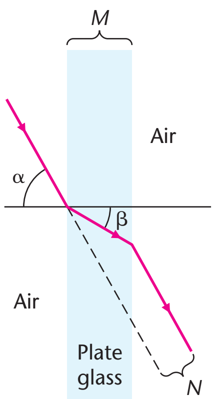
★★**77.** LIGHT REFRACTION Light rays passing through a plate glass window are refracted when they enter the glass and again when they leave to continue on a path parallel to the entering rays (see the figure). If the plate glass is $M$ inches thick, the parallel displacement of the light rays is $N$ inches, the angle of incidence is $\alpha$, and the angle of refraction is $\beta$, show that

$$\tan \beta = \tan \alpha - \frac{N}{M} \sec \alpha$$

[*Hint:* First use geometric relationships to obtain

$$\frac{M}{\sec (90° - \beta)} = \frac{N}{\sin (\alpha - \beta)}$$

then use difference identities and fundamental identities to complete the derivation.]




**78.** LIGHT REFRACTION Use the results of Problem 77 to find $\beta$ to the nearest degree if $\alpha = 43°$, $M = 0.25$ inch, and $N = 0.11$ inch.

★**79.** SURVEYING El Capitan is a large monolithic granite peak that rises straight up from the floor of Yosemite Valley in Yosemite National Park. It attracts rock climbers worldwide. At certain times, the reflection of the peak can be seen in the Merced River that runs along the valley floor. How can the height $H$ of El Capitan above the river be determined by using only a sextant $h$ feet high to measure the angle of elevation, $\beta$, to the top of the peak, and the angle of depression, $\alpha$, of the reflected peak top in the river? (See accompanying figure, which is not to scale.)

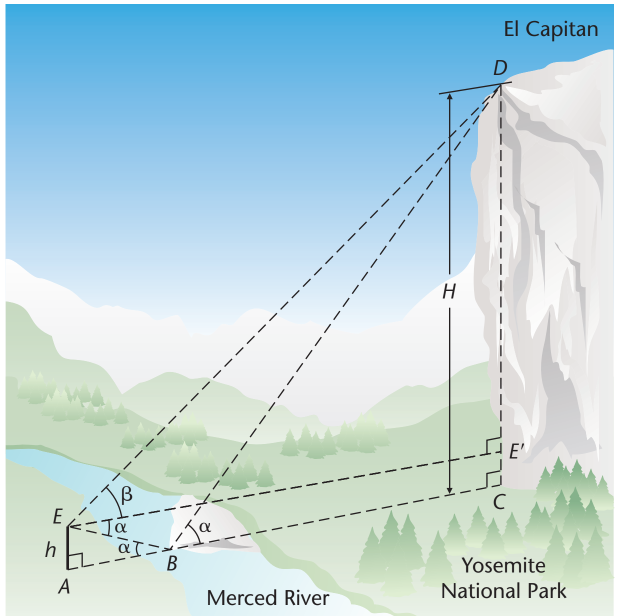
(A) Using right triangle relationships, show that

$$H = h\left(\frac{1 + \tan \beta \cot \alpha}{1 - \tan \beta \cot \alpha}\right)$$

(B) Using sum or difference identities, show that the result in part A can be written in the form

$$H = h\left[\frac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)}\right]$$

(C) If a sextant of height 4.90 feet measures $\alpha$ to be 46.23° and $\beta$ to be 46.15°, compute the height $H$ of El Capitan above the Merced River to three significant digits.




# 7-3 Double-Angle and Half-Angle Identities

- Double-Angle Identities
- Half-Angle Identities

Section 7-3 develops another important set of identities called *double-angle* and *half-angle identities*. We can derive these identities directly from the sum and difference identities given in Section 7-2. Although the names use the word *angle,* the new identities hold for real numbers as well.

## Double-Angle Identities

Start with the sum identity for sine,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

and replace $y$ with $x$ to obtain

$$\sin (x + x) = \sin x \cos x + \cos x \sin x$$

On simplification, this gives

$$\sin 2x = 2 \sin x \cos x \qquad \text{Double-angle identity for sine} \tag{1}$$

If we start with the sum identity for cosine,

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

and replace $y$ with $x$, we obtain

$$\cos (x + x) = \cos x \cos x - \sin x \sin x$$

On simplification, this gives

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \text{First double-angle identity for cosine} \tag{2}$$

Now, using the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1 \tag{3}$$

in the form

$$\cos^2 x = 1 - \sin^2 x \tag{4}$$

and substituting it into equation (2), we get

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

On simplification, this gives

$$\cos 2x = 1 - 2\sin^2 x \qquad \text{Second double-angle identity for cosine} \qquad (5)$$

Or, if we use equation (3) in the form

$$\sin^2 x = 1 - \cos^2 x$$

and substitute it into equation (2), we get

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

On simplification, this gives

$$\cos 2x = 2\cos^2 x - 1 \qquad \text{Third double-angle identity for cosine} \qquad (6)$$

Double-angle identities can be established for the tangent function in the same way by starting with the sum formula for tangent (a good exercise for you).

We list the double-angle identities below for convenient reference.

### › DOUBLE-ANGLE IDENTITIES

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

The identities in the second row can be solved for $\sin^2 x$ and $\cos^2 x$ to obtain the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

These are useful in calculus to transform a power form to a nonpower form.

### ››› EXPLORE-DISCUSS 1

Discuss how you would show that

$$\sin 2x = 2\sin x \qquad \text{is } \textit{not} \text{ an identity.}$$

## EXAMPLE 1 Identity Verification

Verify the identity $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

VERIFICATION

We start with the right side and use quotient identities:

$$\frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$$ Multiply numerator and denominator by $\cos^2 x$.

$$= \frac{\cos^2 x\left(1-\dfrac{\sin^2 x}{\cos^2 x}\right)}{\cos^2 x\left(1+\dfrac{\sin^2 x}{\cos^2 x}\right)}$$ Use algebra to write the compound fraction as a simple fraction.

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$$ Use $\sin^2 x + \cos^2 x = 1$.

$$= \cos^2 x - \sin^2 x$$ Use a double-angle identity.

$$= \cos 2x$$

KEY ALGEBRAIC STEPS IN EXAMPLE 1

$$\frac{1-\dfrac{a^2}{b^2}}{1+\dfrac{a^2}{b^2}} = \frac{b^2\left(1-\dfrac{a^2}{b^2}\right)}{b^2\left(1+\dfrac{a^2}{b^2}\right)} = \frac{b^2-a^2}{b^2+a^2}$$

## MATCHED PROBLEM 1

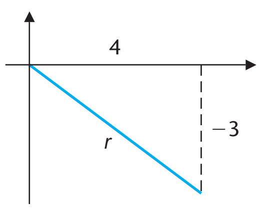
Verify the identity $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$.

## EXAMPLE 2 Finding Exact Values

Find the exact values, without using a calculator, of $\sin 2x$ and $\cos 2x$ if $\tan x = -\frac{3}{4}$ and $x$ is a quadrant IV angle.

SOLUTION

First draw the reference triangle for $x$ and find any unknown sides:




$$r = \sqrt{(-3)^2 + 4^2} = 5$$
$$\sin x = -\tfrac{3}{5}$$
$$\cos x = \tfrac{4}{5}$$

Now use double-angle identities for sine and cosine:

$$\sin 2x = 2\sin x\cos x = 2\left(-\tfrac{3}{5}\right)\left(\tfrac{4}{5}\right) = -\tfrac{24}{25}$$
$$\cos 2x = 2\cos^2 x - 1 = 2\left(\tfrac{4}{5}\right)^2 - 1 = \tfrac{7}{25}$$

**MATCHED PROBLEM 2**

Find the exact values, without using a calculator, of $\cos 2x$ and $\tan 2x$ if $\sin x = \frac{4}{5}$ and $x$ is a quadrant II angle.

## Half-Angle Identities

Half-angle identities are simply double-angle identities stated in an alternate form. Let's start with the double-angle identity for cosine in the form

$$\cos 2m = 1 - 2\sin^2 m$$

Now replace $m$ with $x/2$ and solve for $\sin(x/2)$ [if $2m$ is twice $m$, then $m$ is half of $2m$—think about this]:

$$\cos x = 1 - 2\sin^2\frac{x}{2}$$
$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$
$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \text{Half-angle identity for sine} \qquad (7)$$

where the choice of the sign is determined by the quadrant in which $x/2$ lies.

To obtain a half-angle identity for cosine, start with the double-angle identity for cosine in the form

$$\cos 2m = 2\cos^2 m - 1$$

and let $m = x/2$ to obtain

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \text{Half-angle identity for cosine} \qquad (8)$$

where the sign is determined by the quadrant in which $x/2$ lies.

To obtain a *half-angle identity for tangent,* use the quotient identity and the half-angle formulas for sine and cosine:

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\pm\sqrt{\frac{1-\cos x}{2}}}{\pm\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Thus,

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \text{Half-angle identity for tangent} \tag{9}$$

where the sign is determined by the quadrant in which $x/2$ lies.

Simpler versions of equation (9) can be obtained as follows:

$$\begin{aligned}
\left|\tan\frac{x}{2}\right| &= \sqrt{\frac{1-\cos x}{1+\cos x}} && \text{Multiply radicand by } \frac{1+\cos x}{1+\cos x}. \qquad (10)\\
&= \sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1+\cos x}{1+\cos x}} && \text{Use algebra.}\\
&= \sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}} && \text{Use } \sin^2 x + \cos^2 x = 1.\\
&= \sqrt{\frac{\sin^2 x}{(1+\cos x)^2}} && \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\\
&= \frac{\sqrt{\sin^2 x}}{\sqrt{(1+\cos x)^2}} && \sqrt{\sin^2 x} = |\sin x| \text{ and } \sqrt{(1+\cos x)^2} = 1+\cos x, \text{ because } 1+\cos x \text{ is never negative.}\\
&= \frac{|\sin x|}{1+\cos x}
\end{aligned}$$

All absolute value signs can be dropped, because it can be shown that tan $(x/2)$ and sin $x$ always have the same sign (a good exercise for you). Thus,

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \text{Half-angle identity for tangent} \tag{11}$$

By multiplying the numerator and the denominator in the radicand in equation (10) by $1 - \cos x$ and reasoning as before, we also can obtain

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} \qquad \text{Half-angle identity for tangent} \tag{12}$$

We now list all the half-angle identities for convenient reference.

## HALF-ANGLE IDENTITIES

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

where the sign is determined by the quadrant in which $x/2$ lies.

### ››› EXPLORE-DISCUSS 2

Discuss how you would show that

$$\cos\frac{x}{2} = \frac{1}{2}\cos x \qquad \text{is } \textit{not} \text{ an identity.}$$

How many solutions does the equation have? Explain.

## EXAMPLE 3 Finding Exact Values

Compute the exact value of sin 165° without a calculator using a half-angle identity.

**SOLUTION**

$$\sin 165° = \sin\frac{330°}{2}$$ Use half-angle identity for sine with a positive radical, because sin 165° is positive.

$$= \sqrt{\frac{1-\cos 330°}{2}}$$ $\cos 330° = \frac{\sqrt{3}}{2}$

$$= \sqrt{\frac{1-(\sqrt{3}/2)}{2}}$$ Multiply numerator and denominator of radicand by 2 and simplify.

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

## MATCHED PROBLEM 3

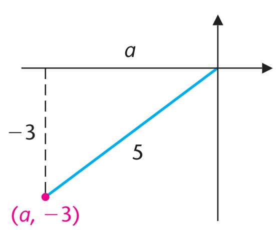
Compute the exact value of tan 105° without a calculator using a half-angle identity.

## EXAMPLE 4 Finding Exact Values

Find the exact values of cos $(x/2)$ and cot $(x/2)$ without using a calculator if $\sin x = -\frac{3}{5}$, $\pi < x < 3\pi/2$.

**SOLUTION**

Draw a reference triangle in the third quadrant, and find cos $x$. Then use appropriate half-angle identities.




$$a = -\sqrt{5^2 - (-3)^2} = -4$$
$$\cos x = -\tfrac{4}{5}$$

If $\pi < x < 3\pi/2$, then

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$ Divide each member of $\pi < x < 3\pi/2$ by 2.

Thus, $x/2$ is an angle in the second quadrant where cosine and cotangent are negative, and

$$\begin{aligned}\cos\frac{x}{2} &= -\sqrt{\frac{1+\cos x}{2}} \\ &= -\sqrt{\frac{1+(-\frac{4}{5})}{2}} \\ &= -\sqrt{\frac{1}{10}} \text{ or } \frac{-\sqrt{10}}{10}\end{aligned}$$

$$\begin{aligned}\cot\frac{x}{2} &= \frac{1}{\tan(x/2)} = \frac{\sin x}{1-\cos x} \\ &= \frac{-\frac{3}{5}}{1-(-\frac{4}{5})} = -\frac{1}{3}\end{aligned}$$

## MATCHED PROBLEM 4

Find the exact values of $\sin(x/2)$ and $\tan(x/2)$ without using a calculator if $\cot x = -\frac{4}{3}$, $\pi/2 < x < \pi$.

## EXAMPLE 5 Identity Verification

Verify the identity $\sin^2\frac{x}{2} = \dfrac{\tan x - \sin x}{2\tan x}$.

### VERIFICATION

We start with the left-hand side and use the half-angle identity for sine to rewrite $\sin\frac{x}{2}$:

$$\sin^2\frac{x}{2} = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2$$ Use algebra.

$$= \frac{1-\cos x}{2}$$ Multiply by $\frac{\tan x}{\tan x}$.

$$= \frac{\tan x}{\tan x}\cdot\frac{1-\cos x}{2}$$ Use algebra.

$$= \frac{\tan x - \tan x\cos x}{2\tan x}$$ Use a quotient identity.

$$= \frac{\tan x - \left(\frac{\sin x}{\cos x}\right)\cos x}{2\tan x}$$ * Simplify.

$$= \frac{\tan x - \sin x}{2\tan x}$$

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

## MATCHED PROBLEM 5

Verify the identity $\cos^2 \dfrac{x}{2} = \dfrac{\tan x + \sin x}{2 \tan x}$.

## ANSWERS TO MATCHED PROBLEMS

**1.** $\dfrac{2 \tan x}{1 + \tan^2 x} = \dfrac{2\left(\dfrac{\sin x}{\cos x}\right)}{1 + \dfrac{\sin^2 x}{\cos^2 x}} = \dfrac{\cos^2 x\left[2\left(\dfrac{\sin x}{\cos x}\right)\right]}{\cos^2 x\left(1 + \dfrac{\sin^2 x}{\cos^2 x}\right)} = \dfrac{2 \sin x \cos x}{\cos^2 x + \sin^2 x}$

$= 2 \sin x \cos x = \sin 2x$

**2.** $\cos 2x = -\frac{7}{25}$, $\tan 2x = \frac{24}{7}$ **3.** $-\sqrt{3} - 2$

**4.** $\sin (x/2) = 3\sqrt{10}/10$, $\tan (x/2) = 3$

**5.** $\cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2} = \dfrac{\tan x}{\tan x} \cdot \dfrac{1 + \cos x}{2} = \dfrac{\tan x + \tan x \cos x}{2 \tan x} = \dfrac{\tan x + \sin x}{2 \tan x}$

# 7-3 Exercises

*In Problems 1–6, verify each identity for the values indicated.*

**1.** $\cos 2x = \cos^2 x - \sin^2 x$, $x = 30°$

**2.** $\sin 2x = 2 \sin x \cos x$, $x = 45°$

**3.** $\tan 2x = \dfrac{2}{\cot x - \tan x}$, $x = \dfrac{\pi}{3}$

**4.** $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$, $x = \dfrac{\pi}{6}$

**5.** $\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$, $x = \pi$
(Choose the correct sign.)

**6.** $\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$, $x = \dfrac{\pi}{2}$
(Choose the correct sign.)

*In Problems 7–10, find the exact value without a calculator using double-angle and half-angle identities.*

**7.** $\sin 22.5°$ **8.** $\tan 75°$

**9.** $\cos 67.5°$ **10.** $\tan 15°$

*In Problems 11–14, graph $y_1$ and $y_2$ in the same viewing window for $-2\pi \le x \le 2\pi$. Use TRACE to compare the two graphs.*

**11.** $y_1 = \cos 2x$, $y_2 = \cos^2 x - \sin^2 x$

**12.** $y_1 = \sin 2x$, $y_2 = 2 \sin x \cos x$

**13.** $y_1 = \tan \dfrac{x}{2}$, $y_2 = \dfrac{\sin x}{1 + \cos x}$

**14.** $y_1 = \tan 2x$, $y_2 = \dfrac{2 \tan x}{1 - \tan^2 x}$

*Verify the identities in Problems 15–32.*

**15.** $(\sin x + \cos x)^2 = 1 + \sin 2x$

**16.** $\sin 2x = (\tan x)(1 + \cos 2x)$

**17.** $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ **18.** $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

**19.** $1 - \cos 2x = \tan x \sin 2x$

**20.** $1 + \sin 2t = (\sin t + \cos t)^2$

**21.** $\sin^2 \dfrac{x}{2} = \dfrac{1 - \cos x}{2}$ **22.** $\cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2}$

**23.** $\cot 2x = \dfrac{1 - \tan^2 x}{2 \tan x}$

**24.** $\cot 2x = \dfrac{\cot x - \tan x}{2}$

**25.** $\cot \dfrac{\theta}{2} = \dfrac{\sin \theta}{1 - \cos \theta}$

**26.** $\cot \dfrac{\theta}{2} = \dfrac{1 + \cos \theta}{\sin \theta}$

**27.** $\cos 2u = \dfrac{1 - \tan^2 u}{1 + \tan^2 u}$

**28.** $\dfrac{\cos 2u}{1 - \sin 2u} = \dfrac{1 + \tan u}{1 - \tan u}$

**29.** $2 \csc 2x = \dfrac{1 + \tan^2 x}{\tan x}$

**30.** $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$

**31.** $\cos \alpha = \dfrac{1 - \tan^2 (\alpha/2)}{1 + \tan^2 (\alpha/2)}$

**32.** $\cos 2\alpha = \dfrac{\cot \alpha - \tan \alpha}{\cot \alpha + \tan \alpha}$

*In Problems 33–40, show that the equation is not an identity by finding a value of x for which both sides are defined but are not equal.*

**33.** $\tan 2x = 2 \tan x$

**34.** $\cos 2x = 2 \cos x$

**35.** $\sin \dfrac{x}{2} = \dfrac{1}{2} \sin x$

**36.** $\tan \dfrac{x}{2} = \dfrac{1}{2} \tan x$

**37.** $\cos \dfrac{x}{2} = \sqrt{\dfrac{1 + \cos x}{2}}$

**38.** $\sin \dfrac{x}{2} = \sqrt{\dfrac{1 - \cos x}{2}}$

**39.** $\tan \dfrac{x}{2} = \sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

**40.** $\tan 2x = \dfrac{2 \cot x}{1 - \cot^2 x}$

*In Problems 41–46, is the equation an identity? Explain.*

**41.** $\sin 4x = 4 \sin x \cos x$

**42.** $\csc 2x = 2 \csc x \sec x$

**43.** $\cot 2x = \dfrac{\tan x (\cot^2 x - 1)}{2}$

**44.** $\tan 4x = 4 \tan x$

**45.** $\cos 2x = 1 - 2 \cos^2 x$

**46.** $\tan 2x = \dfrac{2}{\tan x - \cot x}$

*Compute the exact values of sin 2x, cos 2x, and tan 2x using the information given in Problems 47–50 and appropriate identities. Do not use a calculator.*

**47.** $\sin x = \frac{3}{5}$, $\pi/2 < x < \pi$

**48.** $\cos x = -\frac{4}{5}$, $\pi/2 < x < \pi$

**49.** $\tan x = -\frac{5}{12}$, $-\pi/2 < x < 0$

**50.** $\cot x = -\frac{5}{12}$, $-\pi/2 < x < 0$

*In Problems 51–54, compute the exact values of sin (x/2), cos (x/2), and tan (x/2) using the information given and appropriate identities. Do not use a calculator.*

**51.** $\sin x = -\frac{1}{3}$, $\pi < x < 3\pi/2$

**52.** $\cos x = -\frac{1}{4}$, $\pi < x < 3\pi/2$

**53.** $\cot x = \frac{3}{4}$, $-\pi < x < -\pi/2$

**54.** $\tan x = \frac{3}{4}$, $-\pi < x < -\pi/2$

*Suppose you are tutoring a student who is having difficulties in finding the exact values of sin θ and cos θ from the information given in Problems 55 and 56. Assuming you have worked through each problem and have identified the key steps in the solution process, proceed with your tutoring by guiding the student through the solution process using the following questions. Record the expected correct responses from the student.*

*(A) The angle 2θ is in what quadrant and how do you know?*

*(B) How can you find sin 2θ and cos 2θ? Find each.*

*(C) What identities relate sin θ and cos θ with either sin 2θ or cos 2θ?*

*(D) How would you use the identities in part C to find sin θ and cos θ exactly, including the correct sign?*

*(E) What are the exact values for sin θ and cos θ?*

**55.** Find the exact values of $\sin \theta$ and $\cos \theta$, given $\tan 2\theta = -\frac{4}{3}$, $0° < \theta < 90°$.

**56.** Find the exact values of $\sin \theta$ and $\cos \theta$, given $\sec 2\theta = -\frac{5}{4}$, $0° < \theta < 90°$.

*Verify each of the following identities for the value of x indicated in Problems 57–60. Compute values to five significant digits using a calculator.*

*(A)* $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$ *(B)* $\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

*(Choose the correct sign.)*

**57.** $x = 252.06°$

**58.** $x = 72.358°$

**59.** $x = 0.934\ 57$

**60.** $x = 4$

*In Problems 61–64, graph $y_1$ and $y_2$ in the same viewing window for $-2\pi \le x \le 2\pi$, and state the intervals for which the equation $y_1 = y_2$ is an identity.*

**61.** $y_1 = \cos (x/2)$, $y_2 = \sqrt{\dfrac{1 + \cos x}{2}}$

**62.** $y_1 = \cos (x/2)$, $y_2 = -\sqrt{\dfrac{1 + \cos x}{2}}$

**63.** $y_1 = \sin (x/2)$, $y_2 = -\sqrt{\dfrac{1 - \cos x}{2}}$

**64.** $y_1 = \sin (x/2)$, $y_2 = \sqrt{\dfrac{1 - \cos x}{2}}$

*Verify the identities in Problems 65–68.*

**65.** $\cos 3x = 4\cos^3 x - 3\cos x$

**66.** $\sin 3x = 3\sin x - 4\sin^3 x$

**67.** $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

**68.** $\sin 4x = (\cos x)(4\sin x - 8\sin^3 x)$

*In Problems 69–74, find the exact value of each without using a calculator.*

**69.** $\cos\,[2\cos^{-1}(\frac{3}{5})]$

**70.** $\sin\,[2\cos^{-1}(\frac{3}{5})]$

**71.** $\tan\,[2\cos^{-1}(-\frac{4}{5})]$

**72.** $\tan\,[2\tan^{-1}(-\frac{3}{4})]$

**73.** $\cos\,[\frac{1}{2}\cos^{-1}(-\frac{3}{5})]$

**74.** $\sin\,[\frac{1}{2}\tan^{-1}(-\frac{4}{3})]$

*In Problems 75–80, graph f(x) in a graphing calculator, find a simpler function g(x) that has the same graph as f(x), and verify the identity f(x) = g(x). [Assume g(x) = k + A T(Bx) where k, A, and B are constants and T(x) is one of the six trigonometric functions.]*

**75.** $f(x) = \csc x - \cot x$

**76.** $f(x) = \csc x + \cot x$

**77.** $f(x) = \dfrac{1 - 2\cos 2x}{2\sin x - 1}$

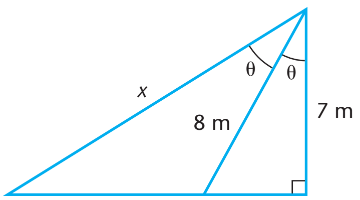
**78.** $f(x) = \dfrac{1 + 2\cos 2x}{1 + 2\cos x}$

**79.** $f(x) = \dfrac{1}{\cot x \sin 2x - 1}$

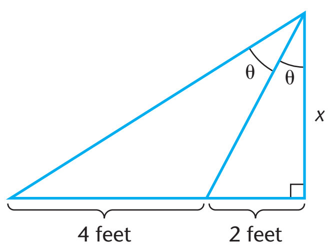
**80.** $f(x) = \dfrac{\cot x}{1 + \cos 2x}$

## APPLICATIONS

★**81. INDIRECT MEASUREMENT** Find the exact value of $x$ in the figure; then find $x$ and $\theta$ to three decimal places. [*Hint:* Use $\cos 2\theta = 2\cos^2\theta - 1$.]




★**82. INDIRECT MEASUREMENT** Find the exact value of $x$ in the figure; then find $x$ and $\theta$ to three decimal places. [*Hint:* Use $\tan 2\theta = (2\tan\theta)/(1 - \tan^2\theta)$.]



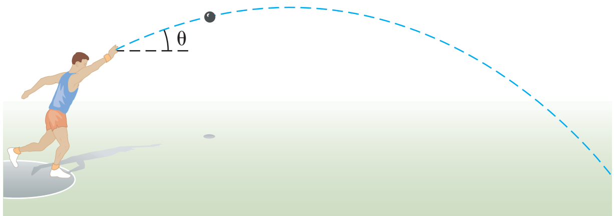

★**83. SPORTS—PHYSICS** The theoretical distance $d$ that a shot-putter, discus thrower, or javelin thrower can achieve on a given throw is found in physics to be given approximately by

$$d = \frac{2v_0^2 \sin\theta\cos\theta}{32 \text{ feet per second per second}}$$

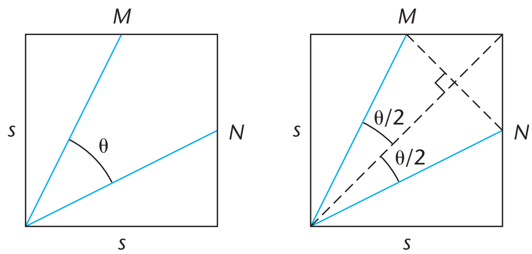
where $v_0$ is the initial speed of the object thrown (in feet per second) and $\theta$ is the angle above the horizontal at which the object leaves the hand (see the figure).

(A) Write the formula in terms of $\sin 2\theta$ by using a suitable identity.

(B) Using the resulting equation in part A, determine the angle $\theta$ that will produce the maximum distance $d$ for a given initial speed $v_0$. This result is an important consideration for shot-putters, javelin throwers, and discus throwers.




★★**84. GEOMETRY** In part (a) of the figure, $M$ and $N$ are the midpoints of the sides of a square. Find the exact value of $\cos\theta$. [*Hint:* The solution uses the Pythagorean theorem, the definition of sine and cosine, a half-angle identity, and some auxiliary lines as drawn in part (b) of the figure.]




(a) (b)

**85. AREA** An $n$-sided regular polygon is inscribed in a circle of radius $R$.

(A) Show that the area of the $n$-sided polygon is given by

$$A_n = \frac{1}{2} nR^2 \sin\frac{2\pi}{n}$$

[*Hint:* (Area of a triangle) $= (\frac{1}{2})$(base)(altitude). Also, a double-angle identity is useful.]

(B) For a circle of radius 1, complete Table 1, to five decimal places, using the formula in part A:

**Table 1**

| $n$ | 10 | 100 | 1,000 | 10,000 |
|---|---|---|---|---|
| $A_n$ | | | | |

(C) What number does $A_n$ seem to approach as $n$ increases without bound? (What is the area of a circle of radius 1?)
(D) Will $A_n$ exactly equal the area of the circumscribed circle for some sufficiently large $n$? How close can $A_n$ be to the area of the circumscribed circle? [In calculus, the area of the circumscribed circle is called the *limit* of $A_n$ as $n$ increases without bound. In symbols, for a circle of radius 1, we would write $\lim_{n \to \infty} A_n = \pi$. The limit concept is the cornerstone on which calculus is constructed.]

# 7-4 Product–Sum and Sum–Product Identities

- Product–Sum Identities
- Sum–Product Identities

Our work with identities is concluded by developing the *product–sum* and *sum–product identities,* which are easily derived from the sum and difference identities developed in Section 7-2. These identities are used in calculus to convert product forms to more convenient sum forms. They also are used in the study of sound waves in music to convert sum forms to more convenient product forms.

## Product–Sum Identities

First, add left side to left side and right side to right side, the sum and difference identities for sine:

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \sin(x - y) &= \sin x \cos y - \cos x \sin y \\ \hline \sin(x + y) + \sin(x - y) &= 2 \sin x \cos y \end{aligned}$$

or

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

Similarly, by adding or subtracting the appropriate sum and difference identities, we can obtain three other **product–sum identities.** These are listed in the box for convenient reference.

**PRODUCT–SUM IDENTITIES**

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$
$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

### EXAMPLE 1 A Product as a Difference

Write the product $\cos 3t \sin t$ as a sum or difference.

SOLUTION

$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)] \quad \text{Let } x = 3t \text{ and } y = t.$$
$$\cos 3t \sin t = \tfrac{1}{2}[\sin(3t + t) - \sin(3t - t)] \quad \text{Simplify.}$$
$$= \tfrac{1}{2}\sin 4t - \tfrac{1}{2}\sin 2t$$

### MATCHED PROBLEM 1

Write the product $\cos 5\theta \cos 2\theta$ as a sum or difference.

### EXAMPLE 2 Finding Exact Values

Evaluate $\sin 105° \sin 15°$ exactly using an appropriate product–sum identity.

SOLUTION

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)] \quad \text{Let } x = 105° \text{ and } y = 15°.$$
$$\sin 105° \sin 15° = \tfrac{1}{2}[\cos(105° - 15°) - \cos(105° + 15°)] \quad \text{Simplify.}$$
$$= \tfrac{1}{2}[\cos 90° - \cos 120°] \quad \cos 90° = 0,\ \cos 120° = -\tfrac{1}{2}$$
$$= \tfrac{1}{2}[0 - (-\tfrac{1}{2})] = \tfrac{1}{4}$$

### MATCHED PROBLEM 2

Evaluate $\cos 165° \sin 75°$ exactly using an appropriate product–sum identity.

## › Sum–Product Identities

The product–sum identities can be transformed into equivalent forms called **sum–product identities.** These identities are used to express sums and differences involving sines and cosines as products involving sines and cosines. We illustrate the transformation for one identity. The other three identities can be obtained by following similar procedures.

We start with a product–sum identity:

$$\sin \alpha \cos \beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad (1)$$

We would like

$$\alpha + \beta = x$$
$$\alpha - \beta = y$$

Solving this system, we have

$$\alpha = \frac{x + y}{2} \qquad \beta = \frac{x - y}{2} \tag{2}$$

Substituting equation (2) into equation (1) and simplifying, we obtain

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

All four sum–product identities are listed next for convenient reference.

**SUM–PRODUCT IDENTITIES**

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

## EXAMPLE 3 A Difference as a Product

Write the difference $\sin 7\theta - \sin 3\theta$ as a product.

**SOLUTION**

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2} \qquad \text{Let } x = 7\theta \text{ and } y = 3\theta.$$

$$\sin 7\theta - \sin 3\theta = 2 \cos \frac{7\theta + 3\theta}{2} \sin \frac{7\theta - 3\theta}{2} \qquad \text{Simplify.}$$

$$= 2 \cos 5\theta \sin 2\theta$$

**MATCHED PROBLEM 3**

Write the sum $\cos 3t + \cos t$ as a product.

## EXAMPLE 4 Finding Exact Values

Find the exact value of sin 105° − sin 15° using an appropriate sum–product identity.

SOLUTION

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \text{Let } x = 105^\circ \text{ and } y = 15^\circ.$$

$$\sin 105^\circ - \sin 15^\circ = 2\cos\frac{105^\circ + 15^\circ}{2}\sin\frac{105^\circ - 15^\circ}{2} \qquad \text{Simplify.}$$

$$= 2\cos 60^\circ \sin 45^\circ \qquad \cos 60^\circ = \tfrac{1}{2},\ \sin 45^\circ = \tfrac{\sqrt{2}}{2}$$

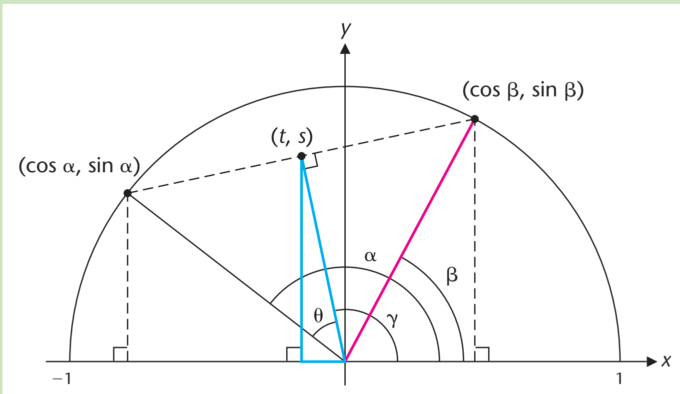
$$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

## MATCHED PROBLEM 4

Find the exact value of cos 165° − cos 75° using an appropriate sum–product identity.

## ››› EXPLORE-DISCUSS 1

The following "proof without words" of two of the sum–product identities is based on a similar "proof" by Sidney H. Kung, Jacksonville University, that was printed in the October 1996 issue of *Mathematics Magazine*. Discuss how the relationships following the figure are verified from the figure.




$$\theta = \frac{\alpha - \beta}{2} \qquad \gamma = \frac{\alpha + \beta}{2}$$

$$\frac{\sin\alpha + \sin\beta}{2} = s = \cos\frac{\alpha - \beta}{2}\sin\frac{\alpha + \beta}{2}$$

$$\frac{\cos\alpha + \cos\beta}{2} = t = \cos\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

**ANSWERS TO MATCHED PROBLEMS**

**1.** $\frac{1}{2}\cos 7\theta + \frac{1}{2}\cos 3\theta$ **2.** $(-\sqrt{3} - 2)/4$ **3.** $2\cos 2t \cos t$ **4.** $-\sqrt{6}/2$

# 7-4 Exercises

*In Problems 1–4, write each product as a sum or difference involving sine and cosine.*

**1.** $\sin 3m \cos m$

**2.** $\cos 7A \cos 5A$

**3.** $\sin u \sin 3u$

**4.** $\cos 2\theta \sin 3\theta$

*In Problems 5–8, write each difference or sum as a product involving sines and cosines.*

**5.** $\sin 3t + \sin t$

**6.** $\cos 7\theta + \cos 5\theta$

**7.** $\cos 5w - \cos 9w$

**8.** $\sin u - \sin 5u$

*Evaluate Problems 9–12 exactly using an appropriate identity.*

**9.** $\sin 195° \cos 75°$

**10.** $\cos 75° \sin 15°$

**11.** $\cos 15° \cos 75°$

**12.** $\sin 105° \sin 165°$

*Evaluate Problems 13–16 exactly using an appropriate identity.*

**13.** $\cos 285° + \cos 195°$

**14.** $\sin 195° + \sin 105°$

**15.** $\cos 15° - \cos 105°$

**16.** $\sin 75° - \sin 165°$

*Use sum and difference identities to verify the identities in Problems 17 and 18.*

**17.** $\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$

**18.** $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

**19.** Explain how you can transform the product–sum identity

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

into the sum–product identity

$$\cos x - \cos y = -2\sin\frac{x + y}{2}\sin\frac{x - y}{2}$$

by a suitable substitution.

**20.** Explain how you can transform the product–sum identity

$$\cos u \cos v = \frac{1}{2}[\cos(u + v) + \cos(u - v)]$$

into the sum–product identity

$$\cos x + \cos y = 2\cos\frac{x + y}{2}\cos\frac{x - y}{2}$$

by a suitable substitution.

*Verify each identity in Problems 21–28.*

**21.** $\dfrac{\sin 2t + \sin 4t}{\cos 2t - \cos 4t} = \cot t$

**22.** $\dfrac{\cos t - \cos 3t}{\sin t + \sin 3t} = \tan t$

**23.** $\dfrac{\sin x - \sin y}{\cos x - \cos y} = -\cot\dfrac{x + y}{2}$

**24.** $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\dfrac{x + y}{2}$

**25.** $\dfrac{\cos x + \cos y}{\sin x - \sin y} = \cot\dfrac{x - y}{2}$

**26.** $\dfrac{\cos x - \cos y}{\sin x + \sin y} = -\tan\dfrac{x - y}{2}$

**27.** $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot\dfrac{x + y}{2}\cot\dfrac{x - y}{2}$

**28.** $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan[\frac{1}{2}(x + y)]}{\tan[\frac{1}{2}(x - y)]}$

*In Problems 29–36, show that the equation is not an identity by finding a value of x and a value of y for which both sides are defined but are not equal.*

**29.** $\sin x \cos y = \sin x + \cos y$

**30.** $\cos x \sin y = \cos x - \sin y$

**31.** $\sin x \sin y = \sin(x + y)$

**32.** $\cos x \cos y = \cos(x + y)$

**33.** $\cos x + \cos y = (\cos x)(\cos y)$

**34.** $\sin x + \sin y = (\sin x)(\sin y)$

**35.** $\sin x - \sin y = \cos \frac{x + y}{2} \sin \frac{x - y}{2}$

**36.** $\cos x - \cos y = 2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$

*In Problems 37–42, is the equation an identity? Explain.*

**37.** $\sin 3x - \sin x = 2 \cos 2x \sin x$

**38.** $2 \sin x \cos 2x = \sin x + \sin 3x$

**39.** $\cos 3x - \cos x = 2 \sin 2x \sin x$

**40.** $2 \cos 3x \cos 5x = \cos 8x + \cos 2x$

**41.** $\cos x + \cos 5x = 2 \cos 2x \cos 3x$

**42.** $2 \sin 4x \cos 2x = \sin 8x + \sin 2x$

*Verify each of the following identities for the values of x and y indicated in Problems 43–46. Evaluate each side to five significant digits.*
*(A)* $\cos x \sin y = \frac{1}{2}[\sin (x + y) - \sin (x - y)]$
*(B)* $\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$

**43.** $x = 172.63°, y = 20.177°$

**44.** $x = 50.137°, y = 18.044°$

**45.** $x = 1.1255, y = 3.6014$

**46.** $x = 0.039\ 17, y = 0.610\ 52$

*In Problems 47–54, write each as a product if y is a sum or difference, or as a sum or difference if y is a product. Enter the original equation in a graphing utility as $y_1$, the converted form as $y_2$, and graph $y_1$ and $y_2$ in the same viewing window. Use TRACE to compare the two graphs.*

**47.** $y = \sin 2x + \sin x$

**48.** $y = \cos 3x + \cos x$

**49.** $y = \cos 1.7x - \cos 0.3x$

**50.** $y = \sin 2.1x - \sin 0.5x$

**51.** $y = \sin 3x \cos x$

**52.** $y = \cos 5x \cos 3x$

**53.** $y = \sin 2.3x \sin 0.7x$

**54.** $y = \cos 1.9x \sin 0.5x$

*Verify each identity in Problems 55 and 56.*

**55.** $\cos x \cos y \cos z = \frac{1}{4}[\cos (x + y - z) + \cos (y + z - x) + \cos (z + x - y) + \cos (x + y + z)]$

**56.** $\sin x \sin y \sin z = \frac{1}{4}[\sin (x + y - z) + \sin (y + z - x) + \sin (z + x - y) - \sin (x + y + z)]$

*In Problems 57–60,*
*(A) Graph $y_1$, $y_2$, and $y_3$ in a graphing utility for $0 \le x \le 1$ and $-2 \le y \le 2$.*
*(B) Convert $y_1$ to a sum or difference and repeat part A.*

**57.** $y_1 = 2 \cos (28\pi x) \cos (2\pi x)$
$y_2 = 2 \cos (2\pi x)$
$y_3 = -2 \cos (2\pi x)$

**58.** $y_1 = 2 \sin (24\pi x) \sin (2\pi x)$
$y_2 = 2 \sin (2\pi x)$
$y_3 = -2 \sin (2\pi x)$

**59.** $y_1 = 2 \sin (20\pi x) \cos (2\pi x)$
$y_2 = 2 \cos (2\pi x)$
$y_3 = -2 \cos (2\pi x)$

**60.** $y_1 = 2 \cos (16\pi x) \sin (2\pi x)$
$y_2 = 2 \sin (2\pi x)$
$y_3 = -2 \sin (2\pi x)$

## APPLICATIONS

*Problems 61 and 62 involve the phenomenon of sound called beats. If two tones having the same loudness and close together in pitch (frequency) are sounded, one following the other, most people have difficulty in differentiating the two tones. However, if the tones are sounded simultaneously, they will interact with each other, producing a low warbling sound called a* ***beat.*** *Musicians, when tuning an instrument with other instruments or a tuning fork, listen for these lower beat frequencies and try to eliminate them by adjusting their instruments. Problems 61 and 62 provide a visual illustration of the beat phenomenon.*

**61.** MUSIC—BEAT FREQUENCIES The equations $y = 0.5 \cos 128\pi t$ and $y = -0.5 \cos 144\pi t$ model sound waves with frequencies 64 and 72 hertz, respectively. If both sounds are emitted simultaneously, a *beat* frequency results.

(A) Show that

$$0.5 \cos 128\pi t - 0.5 \cos 144\pi t = \sin 8\pi t \sin 136\pi t$$

(The product form is more useful to sound engineers.)

(B) Graph each equation in a different viewing window for $0 \le t \le 0.25$:

$y = 0.5 \cos 128\pi t$
$y = -0.5 \cos 144\pi t$
$y = 0.5 \cos 128\pi t - 0.5 \cos 144\pi t$
$y = \sin 8\pi t \sin 136\pi t$

**62.** MUSIC—BEAT FREQUENCIES The equations $y = 0.25 \cos 256\pi t$ and $y = -0.25 \cos 288\pi t$ model sound waves with frequencies 128 and 144 hertz, respectively. If both sounds are emitted simultaneously, a *beat* frequency results.

(A) Show that

$$0.25 \cos 256\pi t - 0.25 \cos 288\pi t = 0.5 \sin 16\pi t \sin 272\pi t$$

(The product form is more useful to sound engineers.)

(B) Graph each equation in a different viewing window for $0 \le t \le 0.125$:

$y = 0.25 \cos 256\pi t$
$y = -0.25 \cos 288\pi t$
$y = 0.25 \cos 256\pi t - 0.25 \cos 288\pi t$
$y = 0.5 \sin 16\pi t \sin 272\pi t$

# 7-5 Trigonometric Equations

- Solving Trigonometric Equations Using an Algebraic Approach
- Solving Trigonometric Equations Using a Graphing Calculator

Sections 7-1 through 7-4 of this chapter consider trigonometric equations called **identities.** These are equations that are true for all replacements of the variable(s) for which both sides are defined. We now consider another class of trigonometric equations, called **conditional equations,** which may be true for some replacements of the variable but false for others. For example,

$$\cos x = \sin x$$

is a conditional equation, because it is true for some values of $x$, for example, $x = \pi/4$, and false for others, such as $x = 0$. (Check both values.)

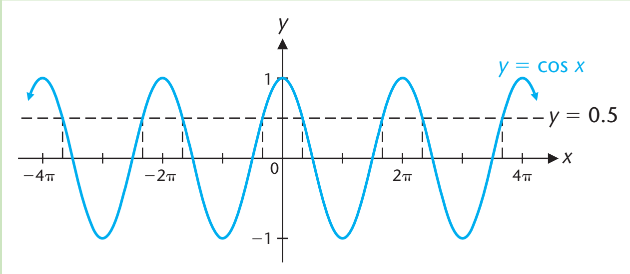
This section considers two approaches for solving conditional trigonometric equations: an algebraic approach and a graphing calculator approach. Solving trigonometric equations using an algebraic approach often requires the use of algebraic manipulation, identities, and ingenuity. In some cases algebraic methods lead to exact solutions, which are very useful in certain contexts. Graphing calculator methods can be used to approximate solutions to a greater variety of trigonometric equations, but usually do not produce exact solutions. Each method has its strengths.

### ››› EXPLORE-DISCUSS 1

We are interested in solutions to the equation

$$\cos x = 0.5$$

The figure shows a partial graph of the left and right sides of the equation.




**(A)** How many solutions does the equation have on the interval $[0, 2\pi)$? What are they?

**(B)** How many solutions does the equation have on the interval $(-\infty, \infty)$? Discuss a method of writing all solutions to the equation.

## › Solving Trigonometric Equations Using an Algebraic Approach

You might find the following suggestions for solving trigonometric equations using an algebraic approach useful:

### › SUGGESTIONS FOR SOLVING TRIGONOMETRIC EQUATIONS ALGEBRAICALLY

**1.** Regard one particular trigonometric function as a variable, and solve for it.

  **(A)** Consider using algebraic manipulation such as factoring, combining or separating fractions, and so on.

  **(B)** Consider using identities.

**2.** After solving for a trigonometric function, solve for the variable.

Examples 1–5 should help make the algebraic approach clear.

### EXAMPLE 1 Exact Solutions Using Factoring

Find all solutions exactly for $2\cos^2 x - \cos x = 0$.

SOLUTION

**Step 1.** *Solve for cos x.*

$$2\cos^2 x - \cos x = 0 \qquad 2a^2 - a = a(2a - 1)$$
$$\cos x\,(2\cos x - 1) = 0 \qquad ab = 0 \text{ only if } a = 0 \text{ or } b = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$\cos x = \tfrac{1}{2}$$

**Step 2.** *Solve each equation over one period [0, 2π).* Sketch a graph of $y = \cos x$, $y = 0$, and $y = \frac{1}{2}$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 1).

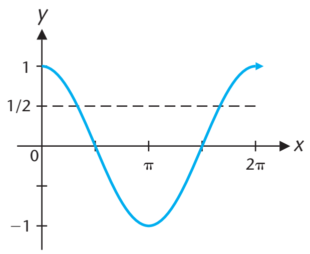



Figure 1

$$\cos x = 0 \qquad\qquad \cos x = \tfrac{1}{2}$$
$$x = \pi/2,\ 3\pi/2 \qquad\qquad x = \pi/3,\ 5\pi/3$$

**Step 3.** *Write an expression for all solutions.* Because the cosine function is periodic with period $2\pi$, all solutions are given by

$$x = \begin{cases} \pi/3 + 2k\pi \\ \pi/2 + 2k\pi \\ 3\pi/2 + 2k\pi \\ 5\pi/3 + 2k\pi \end{cases} \qquad k \text{ any integer}$$

## MATCHED PROBLEM 1

Find all solutions exactly for $2\sin^2 x + \sin x = 0$.

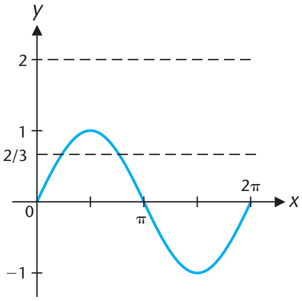
## EXAMPLE 2 Approximate Solutions Using Identities and Factoring

Find all real solutions for $3\cos^2 x + 8\sin x = 7$. Compute all inverse functions to four decimal places.

SOLUTION

**Step 1.** *Solve for sin x and/or cos x.* Move all nonzero terms to the left of the equal sign and express the left side in terms of sin $x$:

$$3\cos^2 x + 8\sin x = 7$$
$$3\cos^2 x + 8\sin x - 7 = 0 \qquad \cos^2 x = 1 - \sin^2 x$$
$$3(1 - \sin^2 x) + 8\sin x - 7 = 0$$
$$3\sin^2 x - 8\sin x + 4 = 0 \qquad 3u^2 - 8u + 4 = (u - 2)(3u - 2)$$
$$(\sin x - 2)(3\sin x - 2) = 0 \qquad ab = 0 \text{ only if } a = 0 \text{ or } b = 0$$
$$\sin x - 2 = 0 \quad \text{or} \quad 3\sin x - 2 = 0$$
$$\sin x = 2 \qquad\qquad \sin x = \tfrac{2}{3}$$




› **Figure 2**

**Step 2.** *Solve each equation over one period [0, 2π):* Sketch a graph of $y = \sin x$, $y = 2$, and $y = \frac{2}{3}$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 2).

Solve the first equation:

$$\sin x = 2$$ No solution, because $+1 \le \sin x \le 1$.

Solve the second equation:

$$\sin x = \tfrac{2}{3}$$ From the graph we see there are solutions in the first and second quadrants.

$$x = \sin^{-1}\tfrac{2}{3} = 0.7297$$ First quadrant solution

$$x = \pi - 0.7297 = 2.4119$$ Second quadrant solution

CHECK

$$\sin 0.7297 = 0.6667; \ \sin 2.4119 = 0.6666$$

(Checks may not be exact because of roundoff errors.)

**Step 3.** *Write an expression for all solutions.* Because the sine function is periodic with period $2\pi$, all solutions are given by

$$x = \begin{cases} 0.7297 + 2k\pi \\ 2.4119 + 2k\pi \end{cases} \quad k \text{ any integer}$$

## MATCHED PROBLEM 2

Find all real solutions to $8 \sin^2 x = 5 - 10 \cos x$. Compute all inverse functions to four decimal places.

## EXAMPLE 3 Approximate Solutions Using Substitution

Find θ in degree measure to three decimal places so that $5 \sin(2\theta - 5) = -3.045$, $0° \le 2\theta - 5 \le 360°$.

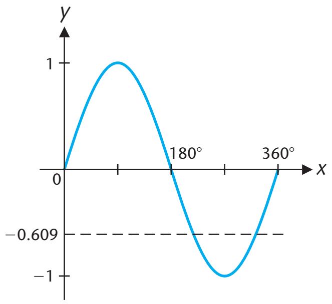
SOLUTION

**Step 1.** *Make a substitution.* Let $u = 2\theta - 5$ to obtain

$$5 \sin u = -3.045, \ 0° \le u \le 360°$$

**Step 2.** *Solve for sin u.*

$$\sin u = \frac{-3.045}{5} = -0.609$$




› Figure 3

**Step 3.** *Solve for u over $0° \le u \le 360°$.* Sketch a graph of $y = \sin u$ and $y = -0.609$ in the same coordinate system to provide an aid to writing all solutions over $0° \le u \le 360°$ (Fig. 3).
Solutions are in the third and fourth quadrants. If the reference angle is $\alpha$, then $u = 180° + \alpha$ or $u = 360° - \alpha$.

$$\alpha = \sin^{-1} 0.609 = 37.517° \quad \text{Reference angle}$$
$$u = 180° + 37.517°$$
$$= 217.517° \quad \text{Third quadrant solution}$$
$$u = 360° - 37.517°$$
$$= 322.483° \quad \text{Fourth quadrant solution}$$

CHECK

$$\sin 217.517° = -0.609; \sin 322.483° = -0.609$$

**Step 4.** *Now solve for* $\theta$:

$$u = 217.517° \qquad u = 322.483°$$
$$2\theta - 5 = 217.517° \qquad 2\theta - 5 = 322.483°$$
$$\boldsymbol{\theta = 111.259°} \qquad \boldsymbol{\theta = 163.742°}$$

A final check in the original equation is left to the reader.

## MATCHED PROBLEM 3

Find $\theta$ in degree measure to three decimal places so that $8 \tan (6\theta + 15) = -64.328$, $-90° < 6\theta + 15 < 90°$.

## EXAMPLE 4 Exact Solutions Using Identities and Factoring

Find exact solutions for $\sin^2 x = \frac{1}{2}\sin 2x$, $0 \le x \le 2\pi$.

SOLUTION

The following solution includes only the key steps. Sketch graphs as appropriate on scratch paper.

$$\sin^2 x = \tfrac{1}{2}\sin 2x \quad \text{Use double-angle identity.}$$
$$= \tfrac{1}{2}(2 \sin x \cos x) \quad \text{Subtract } \sin x \cos x \text{ from both sides.}$$
$$\sin^2 x - \sin x \cos x = 0 \quad a^2 - ab = a(a - b)$$
$$\sin x (\sin x - \cos x) = 0 \quad a(a - b) = 0 \text{ only if } a = 0 \text{ or } a - b = 0$$
$$\sin x = 0 \quad \text{or} \quad \sin x - \cos x = 0$$
$$x = 0, \pi \qquad \sin x = \cos x$$
$$\frac{\sin x}{\cos x} = 1$$
$$\tan x = 1$$
$$x = \pi/4, 5\pi/4$$

Combining the solutions from both equations, we have the complete set of solutions:

$$x = 0, \pi/4, \pi, 5\pi/4$$

## MATCHED PROBLEM 4

Find exact solutions for $\sin 2x = \sin x$, $0 \le x \le 2\pi$.

## EXAMPLE 5 Approximate Solutions Using Identities and the Quadratic Formula

Solve $\cos 2x = 4 \cos x - 2$ for all real $x$. Compute inverse functions to four decimal places.

SOLUTION

**Step 1.** *Solve for cos x.*

$$\cos 2x = 4 \cos x - 2$$ Use double-angle identity.

$$2 \cos^2 x - 1 = 4 \cos x - 2$$ Subtract $4 \cos x - 2$ from both sides.

$$2 \cos^2 x - 4 \cos x + 1 = 0$$ Quadratic in cos $x$. Left side does not factor using integer coefficients. Solve using quadratic formula with $a = 2$, $b = -4$, and $c = 1$.

$$\cos x = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2(2)}$$

$$= 1.707107 \text{ or } 0.292893$$

**Step 2.** *Solve each equation over one period [0, 2π):* Sketch a graph of $y = \cos x$, $y = 1.707107$, and $y = 0.292893$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 4).

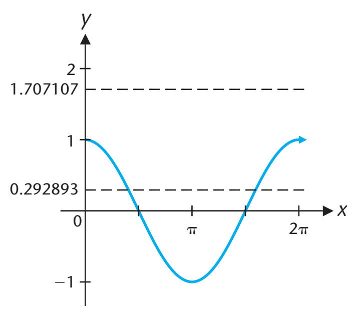



Figure 4

Solve the first equation:

$$\cos x = 1.707107$$ No solution, because $-1 \le \cos x \le 1$

Solve the second equation:

$$\cos x = 0.292893$$

Figure 4 indicates a first quadrant solution and a fourth quadrant solution. If the reference angle is $\alpha$, then $x = \alpha$ or $x = 2\pi - \alpha$.

$$\alpha = \cos^{-1} 0.292893 = 1.2735$$
$$2\pi - \alpha = 2\pi - 1.2735 = 5.0096$$

CHECK

$$\cos 1.2735 = 0.292936; \cos 5.0096 = 0.292854$$

**Step 3.** *Write an expression for all solutions.* Because the cosine function is periodic with period $2\pi$, all solutions are given by

$$x = \begin{cases} 1.2735 + 2k\pi \\ 5.0096 + 2k\pi \end{cases} \quad k \text{ any integer}$$

**MATCHED PROBLEM 5**

Solve $\cos 2x = 2(\sin x - 1)$ for all real $x$. Compute inverse functions to four decimal places.

## Solving Trigonometric Equations Using a Graphing Calculator

All the trigonometric equations that were solved earlier with algebraic methods can also be solved, though usually not exactly, with graphing calculator methods. In addition, there are many trigonometric equations that can be solved (to any decimal accuracy desired) using graphing calculator methods, but cannot be solved in a finite sequence of steps using algebraic methods. Examples 6–8 are examples of such equations.

**EXAMPLE 6** Solution Using a Graphing Calculator

Find all real solutions to four decimal places for $2 \cos 2x = 1.35x - 2$.

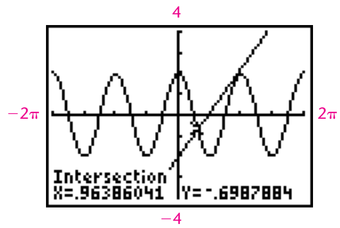
SOLUTION

This relatively simple trigonometric equation cannot be solved using a finite number of algebraic steps (try it!). However, it can be solved rather easily to the accuracy desired using a graphing calculator. Graph $y_1 = 2 \cos 2x$ and $y_2 = 1.35x - 2$ in the same viewing window, and find any points of intersection using the intersect command. The first point of intersection is shown in Figure 5. It appears there may be more than one point of intersection, but zooming in on the portion of the graph in question shows that the two graphs do not intersect in that region (Fig. 6). The only solution is

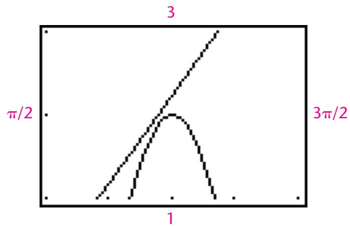
$$x = 0.9639$$

**CHECK**

Left side: $2 \cos 2(0.9639) = -0.6989$
Right side: $1.35(0.9639) - 2 = -0.6987$




› Figure 5




› Figure 6

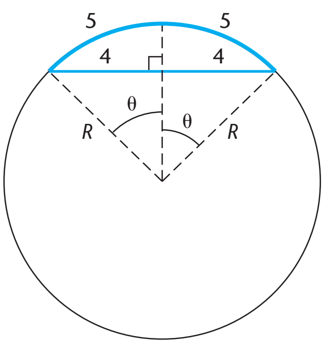
## MATCHED PROBLEM 6

Find all real solutions to four decimal places for $\sin x/2 = 0.2x - 0.5$.

## EXAMPLE 7 Geometric Application

A 10-centimeter arc on a circle has an 8-centimeter chord. What is the radius of the circle to four decimal places? What is the radian measure of the central angle, to four decimal places, subtended by the arc?

**SOLUTION**




› Figure 7

Sketch a figure with auxiliary lines (Fig. 7).

From the figure, θ in radians is

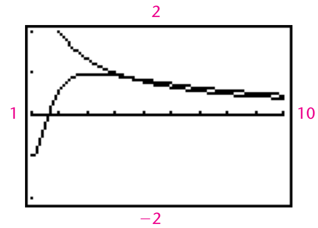
$$\theta = \frac{5}{R} \qquad \text{and} \qquad \sin\theta = \frac{4}{R}$$

Thus,

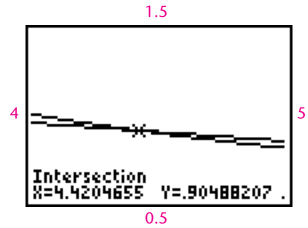
$$\sin\frac{5}{R} = \frac{4}{R}$$

and our problem is to solve this trigonometric equation for $R$. Algebraic methods will not isolate $R$, so instead of using algebra, we turn to the use of a graphing calculator.

Start by graphing $y_1 = \sin 5/x$ and $y_2 = 4/x$ in the same viewing window for $1 \le x \le 10$ and $-2 \le y \le 2$ (Fig. 8). It appears that the graphs intersect for $x$ between 4 and 5. To get a clearer look at the intersection point we change the window dimensions to $4 \le x \le 5$ and $0.5 \le y \le 1.5$, and use the intersect command to find the point of intersection (Fig. 9).



› Figure 8



› Figure 9

From Figure 9, we see that

$$R = 4.4205 \text{ centimeters} \qquad \text{To four decimal places.}$$

CHECK

$$\sin 5/R = \sin (5/4.4205) \approx 0.9049; \; 4/R = 4/4.4205 \approx 0.9049$$

Having $R$, we can compute the radian measure of the central angle subtended by the 10-centimeter arc:

$$\text{Central Angle} = \frac{10}{R} = \frac{10}{4.4205} = 2.2622 \text{ radians} \qquad \text{To four decimal places.}$$

## MATCHED PROBLEM 7

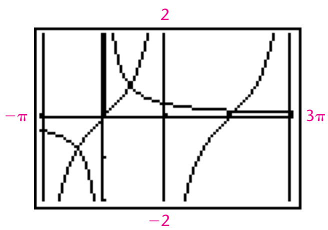
An 8.2456-inch arc on a circle has a 6.0344-inch chord. What is the radius of the circle to four decimal places? What is the measure of the central angle, to four decimal places, subtended by the arc?

## EXAMPLE 8 Solution Using a Graphing Calculator

Find all real solutions, to four decimal places, for $\tan (x/2) = 1/x$, $-\pi < x \le 3\pi$.

SOLUTION

Graph $y_1 = \tan (x/2)$ and $y_2 = 1/x$ in the same viewing window for $-\pi < x < 3\pi$ (Fig. 10). Solutions are at points of intersection.



› Figure 10

Using the intersect command, the three solutions are found to be

$$x = -1.3065, 1.3065, 6.5846$$

Checking these solutions is left to the reader.

## MATCHED PROBLEM 8

Find all real solutions, to four decimal places, for $0.25 \tan (x/2) = \ln x$, $0 < x < 4\pi$.

Solving trigonometric inequalities using a graphing calculator is as easy as solving trigonometric equations using a graphing calculator. Example 9 illustrates the process.

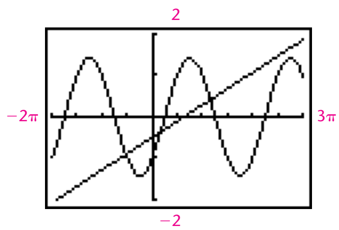
## EXAMPLE 9 Solving a Trigonometric Inequality

Solve $\sin x - \cos x < 0.25x - 0.5$, using two-decimal-place accuracy.

### SOLUTION

Graph $y_1 = \sin x - \cos x$ and $y_2 = 0.25x - 0.5$ in the same viewing window (Fig. 11).

Finding the three points of intersection by the intersect command, we see that the graph of $y_1$ is below the graph of $y_2$ on the following two intervals: $(-1.65, 0.52)$ and $(3.63, \infty)$. Thus, the solution set to the inequality is $(-1.65, 0.52) \cup (3.63, \infty)$.




› **Figure 11**

## MATCHED PROBLEM 9

Solve $\cos x - \sin x > 0.4 - 0.3x$, using two-decimal-place accuracy.

## ››› EXPLORE-DISCUSS 2

How many solutions does the following equation have?

$$\sin (1/x) = 0 \qquad (1)$$

Graph $y_1 = \sin (1/x)$ and $y_2 = 0$ for each of the indicated intervals in parts A–G. From each graph estimate the number of solutions that equation (1) appears to have. What final conjecture would you be willing to make regarding the number of solutions to equation (1)? Explain.

**(A)** $[-20, 20]$; Can 0 be a solution? Explain.

**(B)** $[-2, 2]$ **(C)** $[-1, 1]$ **(D)** $[-0.1, 0.1]$ **(E)** $[-0.01, 0.01]$

**(F)** $[-0.001, 0.001]$ **(G)** $[-0.0001, 0.0001]$

## ANSWERS TO MATCHED PROBLEMS

**1.** $x = \begin{cases} 0 + 2k\pi \\ \pi + 2k\pi \\ 7\pi/6 + 2k\pi \\ 11\pi/6 + 2k\pi \end{cases}$ $k$ any integer **2.** $x = \begin{cases} 1.8235 + 2k\pi \\ 4.4597 + 2k\pi \end{cases}$ $k$ any integer

**3.** $-16.318°$ **4.** $x = 0, \pi/3, \pi, 5\pi/3$ **5.** $x = \begin{cases} 0.9665 + 2k\pi \\ 2.1751 + 2k\pi \end{cases}$ $k$ any integer

**6.** $x = 5.1609$ **7.** $R = 3.1103$ inches; central angle $= 2.6511$ radians

**8.** $x = 1.1828, 2.6369, 9.2004$ **9.** $(-1.67, 0.64) \cup (3.46, \infty)$

# 7-5 Exercises

*In Problems 1–20, find exact solutions over the indicated intervals (x a real number, θ in degrees).*

**1.** $\sin x + 1 = 0, 0 \le x < 2\pi$

**2.** $\cos x - 1 = 0, 0 \le x < 2\pi$

**3.** $\sin x + 1 = 0$, all real $x$

**4.** $\cos x - 1 = 0$, all real $x$

**5.** $\tan \theta - 1 = 0, 0° \le \theta < 360°$

**6.** $\tan \theta + 1 = 0, 0° \le \theta < 360°$

**7.** $\tan \theta - 1 = 0$, all $\theta$

**8.** $\tan \theta + 1 = 0$, all $\theta$

**9.** $2 \sin x - 1 = 0, 0 \le x < 2\pi$

**10.** $2 \cos x - 1 = 0, 0 \le x < 2\pi$

**11.** $2 \sin x - 1 = 0$, all real $x$

**12.** $2 \cos x - 1 = 0$, all real $x$

**13.** $2 \sin \theta + \sqrt{3} = 0, 0° \le \theta < 360°$

**14.** $\sqrt{2} \cos \theta - 1 = 0, 0° \le \theta < 360°$

**15.** $2 \sin \theta + \sqrt{3} = 0$, all $\theta$

**16.** $\sqrt{2} \cos \theta - 1 = 0$, all $\theta$

**17.** $\tan x - \sqrt{3} = 0, 0 \le x < 2\pi$

**18.** $\sqrt{3} \tan x - 1 = 0, 0 \le x < 2\pi$

**19.** $\tan x - \sqrt{3} = 0$, all real $x$

**20.** $\sqrt{3} \tan x - 1 = 0$, all real $x$

*Solve Problems 21–26 to four decimal places (θ in degrees, x real).*

**21.** $7 \cos x - 3 = 0, 0 \le x < 2\pi$

**22.** $5 \cos x - 2 = 0, 0 \le x < 2\pi$

**23.** $2 \tan \theta - 7 = 0, 0° \le \theta < 180°$

**24.** $4 \tan \theta + 15 = 0, 0° \le \theta < 180°$

**25.** $1.3224 \sin x + 0.4732 = 0$, all real $x$

**26.** $5.0118 \sin x - 3.1105 = 0$, all real $x$

*Solve Problems 27–30 to four decimal places using a graphing calculator.*

**27.** $1 - x = 2 \sin x$, all real $x$

**28.** $2x - \cos x = 0$, all real $x$

**29.** $\tan (x/2) = 8 - x, 0 \le x < \pi$

**30.** $\tan 2x = 1 + 3x, 0 \le x < \pi/4$

*In Problems 31–46, find exact solutions for x real and θ in degrees.*

**31.** $2 \sin^2 \theta + \sin 2\theta = 0$, all $\theta$

**32.** $\cos^2 \theta = \frac{1}{2} \sin 2\theta$, all $\theta$

**33.** $\tan x = -2 \sin x, 0 \le x < 2\pi$

**34.** $\cos x = \cot x, 0 \le x < 2\pi$

**35.** $\sec (x/2) + 2 = 0, 0 \le x < 2\pi$

**36.** $\tan(x/2) - 1 = 0, 0 \le x < 2\pi$

**37.** $2\cos^2\theta + 3\sin\theta = 0, 0° \le \theta < 360°$

**38.** $\sin^2\theta + 2\cos\theta = -2, 0° \le \theta < 360°$

**39.** $\cos 2\theta + \cos\theta = 0, 0° \le \theta < 360°$

**40.** $\cos 2\theta + \sin^2\theta = 0, 0° \le \theta < 360°$

**41.** $2\sin^2(x/2) - 3\sin(x/2) + 1 = 0, 0 \le x \le 2\pi$

**42.** $4\cos^2 2x - 4\cos 2x + 1 = 0, 0 \le x \le 2\pi$

**43.** $\cos^2 x + \sin^2 x = 1, 0 \le x < 2\pi$

**44.** $\sin x + \cos x = 3, 0 \le x < 2\pi$

**45.** $2\sin\theta = 5 + \cos\theta, 0° \le \theta < 360°$

**46.** $2\cos^2\theta = 1 + \cos 2\theta, 0° \le \theta < 360°$

*Solve Problems 47–52 (x real and θ in degrees). Compute inverse functions to four significant digits.*

**47.** $6\sin^2\theta + 5\sin\theta = 6, 0° \le \theta \le 90°$

**48.** $4\cos^2\theta = 7\cos\theta + 2, 0° \le \theta \le 180°$

**49.** $3\cos^2 x - 8\cos x = 3, 0 \le x \le \pi$

**50.** $8\sin^2 x + 10\sin x = 3, 0 \le x \le \pi/2$

**51.** $2\sin x = \cos 2x, 0 \le x < 2\pi$

**52.** $\cos 2x + 10\cos x = 5, 0 \le x < 2\pi$.

*Solve Problems 53 and 54 for all real number solutions. Compute inverse functions to four significant digits.*

**53.** $2\sin^2 x = 1 - 2\sin x$

**54.** $\cos^2 x = 3 - 5\cos x$

*Solve Problems 55–64 to four decimal places using a graphing calculator.*

**55.** $2\sin x = \cos 2x, 0 \le x < 2\pi$

**56.** $\cos 2x + 10\cos x = 5, 0 \le x < 2\pi$

**57.** $2\sin^2 x = 1 - 2\sin x$, all real $x$

**58.** $\cos^2 x = 3 - 5\cos x$, all real $x$

**59.** $\cos 2x > x^2 - 2$, all real $x$

**60.** $2\sin(x - 2) < 3 - x^2$, all real $x$

**61.** $\cos(2x + 1) \le 0.5x - 2$, all real $x$

**62.** $\sin(3 - 2x) \ge 1 - 0.4x$, all real $x$

**63.** $e^{\sin x} = 2x - 1$, all real $x$

**64.** $e^{-\sin x} = 3 - x$, all real $x$

**65.** Explain the difference between evaluating $\tan^{-1}(-5.377)$ and solving the equation $\tan x = -5.377$.

**66.** Explain the difference between evaluating $\cos^{-1}(-0.7334)$ and solving the equation $\cos x = -0.7334$.

*Find exact solutions to Problems 67–70. [Hint: Square both sides at an appropriate point, solve, then eliminate extraneous solutions at the end.]*

**67.** $\cos x - \sin x = 1, 0 \le x < 2\pi$

**68.** $\sin x + \cos x = 1, 0 \le x < 2\pi$

**69.** $\tan x - \sec x = 1, 0 \le x < 2\pi$

**70.** $\sec x + \tan x = 1, 0 \le x < 2\pi$

*Solve Problems 71–72 to four significant digits using a graphing utility.*

**71.** $\sin(1/x) = 1.5 - 5x, 0.04 \le x \le 0.2$

**72.** $2\cos(1/x) = 950x - 4, 0.006 < x < 0.007$

**73.** We are interested in the zeros of $f(x) = \sin(1/x)$ for $x > 0$.

(A) Explore the graph of $f$ over different intervals $[0.1, b]$ for various values of $b$, $b > 0.1$. Does the function $f$ have a largest zero? If so, what is it (to four decimal places)? Explain what happens to the graph of $f$ as $x$ increases without bound. Does the graph have an asymptote? If so, what is its equation?

(B) Explore the graph of $f$ over different intervals $(0, b]$ for various values of $b$, $0 < b \le 0.1$. How many zeros exist between 0 and $b$, for any $b > 0$, however small? Explain why this happens. Does $f$ have a smallest positive zero? Explain.

**74.** We are interested in the zeros of $g(x) = \cos(1/x)$ for $x > 0$.

(A) Explore the graph of $g$ over different intervals $[0.1, b]$ for various values of $b$, $b > 0.1$. Does the function $g$ have a largest zero? If so, what is it (to four decimal places)? Explain what happens to the graph of $g$ as $x$ increases without bound. Does the graph have an asymptote? If so, what is its equation?

(B) Explore the graph of $g$ over different intervals $(0, b]$ for various values of $b$, $0 < b \le 0.1$. How many zeros exist between 0 and $b$, for any $b > 0$, however small? Explain why this happens. Does $g$ have a smallest positive zero? Explain.

## APPLICATIONS

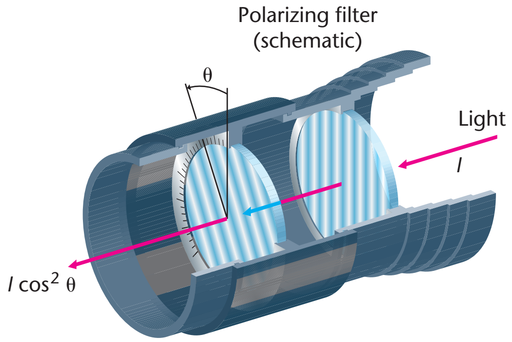
**75. ELECTRIC CURRENT** An alternating current generator produces a current given by the equation

$$I = 30 \sin 120\pi t$$

where $t$ is time in seconds and $I$ is current in amperes. Find the smallest positive $t$ (to four significant digits) such that $I = -10$ amperes.

**76. ELECTRIC CURRENT** Refer to Problem 75. Find the smallest positive $t$ (to four significant digits) such that $I = 25$ amperes.

**77. OPTICS** A polarizing filter for a camera contains two parallel plates of polarizing glass, one fixed and the other able to rotate. If $\theta$ is the angle of rotation from the position of maximum light transmission, then the intensity of light leaving the filter is $\cos^2 \theta$ times the intensity $I$ of light entering the filter (see the figure).



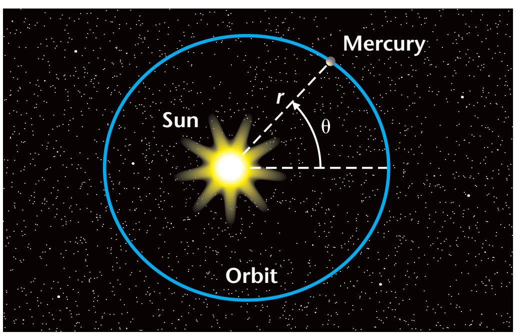

Find the smallest positive $\theta$ (in decimal degrees to two decimal places) so that the intensity of light leaving the filter is 40% of that entering.

**78. OPTICS** Refer to Problem 77. Find the smallest positive $\theta$ so that the light leaving the filter is 70% of that entering.

**79. ASTRONOMY** The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.44 \times 10^7}{1 - 0.206 \cos \theta}$$

(see the figure). Find the smallest positive $\theta$ (in decimal degrees to three significant digits) such that Mercury is $3.09 \times 10^7$ miles from the sun.

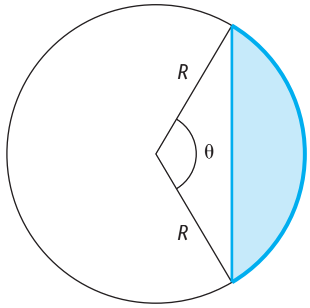



**80. ASTRONOMY** Refer to Problem 79. Find the smallest positive $\theta$ (in decimal degrees to three significant digits) such that Mercury is $3.78 \times 10^7$ miles from the sun.

**81. GEOMETRY** The area of the segment of a circle in the figure is given by

$$A = \tfrac{1}{2}R^2(\theta - \sin \theta)$$

where $\theta$ is in radian measure. Use a graphing calculator to find the radian measure, to three decimal places, of angle $\theta$, if the radius is 8 inches and the area of the segment is 48 square inches.

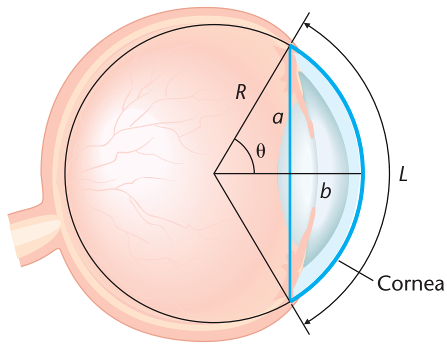



**82. GEOMETRY** Repeat Problem 81, if the radius is 10 centimeters and the area of the segment is 40 square centimeters.

**83. EYE SURGERY** A surgical technique for correcting an astigmatism involves removing small pieces of tissue to change the curvature of the cornea.* In the cross section of a cornea shown in the figure, the circular arc, with radius $R$ and central angle $2\theta$, represents a cross section of the surface of the cornea.

(A) If $a = 5.5$ millimeters and $b = 2.5$ millimeters, find $L$ correct to four decimal places.

(B) Reducing the chord length $2a$ without changing the length $L$ of the arc has the effect of pushing the cornea outward and giving it a rounder, yet still a circular, shape. With the aid of a graphing utility in part of the solution, approximate $b$ to four decimal places if $a$ is reduced to 5.4 millimeters and $L$ remains the same as it was in part A.




*Based on the article "The Surgical Correction of Astigmatism" by Sheldon Rothman and Helen Strassberg in the *UMAP Journal,* Vol. v, no. 2, 1984.

**84.** EYE SURGERY Refer to Problem 83.
(A) If in the figure $a = 5.4$ millimeters and $b = 2.4$ millimeters, find $L$ correct to four decimal places.
(B) Increasing the chord length without changing the arc length $L$ has the effect of pulling the cornea inward and giving it a flatter, yet still circular, shape. With the aid of a graphing utility in part of the solution, approximate $b$ to four decimal places if $a$ is increased to 5.5 millimeters and $L$ remains the same as it was in part A.

ANALYTIC GEOMETRY *Find simultaneous solutions for each system of equations in Problems 85 and 86 ($0° \le \theta \le 360°$). These are polar equations, which will be discussed in Chapter 8.*

★**85.** $r = 2 \sin \theta$
$r = \sin 2\theta$

★**86.** $r = 2 \sin \theta$
$r = 2(1 - \sin \theta)$

*Problems 87 and 88 are related to rotation of axes in analytic geometry.*

★★**87.** ANALYTIC GEOMETRY Given the equation $2xy = 1$, replace $x$ and $y$ with

$$x = u \cos \theta - v \sin \theta$$
$$y = u \sin \theta + v \cos \theta$$

and simplify the left side of the resulting equation. Find the smallest positive $\theta$ in degree measure so that the coefficient of the $uv$ term is 0.

★★**88.** ANALYTIC GEOMETRY Repeat Problem 87 for $xy = -2$.

# CHAPTER 7 Review

## 7-1 Basic Identities and Their Use

The following 11 identities are basic to the process of changing trigonometric expressions to equivalent but more useful forms:

**Reciprocal Identities**

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

**Quotient Identities**

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

**Identities for Negatives**

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

**Pythagorean Identities**

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Although there is no fixed method of verification that works for all identities, the following suggested steps are helpful in many cases.

**Suggested Steps in Verifying Identities**

**1.** Start with the more complicated side of the identity, and transform it into the simpler side.

**2.** Try algebraic operations such as multiplying, factoring, combining fractions, and splitting fractions.

**3.** If other steps fail, express each function in terms of sine and cosine functions, and then perform appropriate algebraic operations.

**4.** At each step, keep the other side of the identity in mind. This often reveals what you should do to get there.

## 7-2 Sum, Difference, and Cofunction Identities

**Sum Identities**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

**Difference Identities**

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

**Cofunction Identities**

(Replace $\pi/2$ with 90° if $x$ is in degrees.)

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

## 7-3 Double-Angle and Half-Angle Identities

**Double-Angle Identities**

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

**Half-Angle Identities**

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

## 7-4 Product–Sum and Sum–Product Identities

**Product–Sum Identities**

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

**Sum–Product Identities**

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

## 7-5 Trigonometric Equations

Sections 7-1 through 7-4 of the chapter considered trigonometric equations called **identities.** Identities are true for all replacements of the variable(s) for which both sides are defined. Section 7-5 considered **conditional equations.** Conditional equations may be true for some variable replacements, but are false for other variable replacements for which both sides are defined. The equation $\sin x = \cos x$ is a conditional equation.

In *solving a trigonometric equation using an algebraic approach,* no particular rule will always lead to all solutions of every trigonometric equation you are likely to encounter. Solving trigonometric equations algebraically often requires the use of algebraic manipulation, identities, and ingenuity.

**Suggestions for Solving Trigonometric Equations Algebraically**

1. Regard one particular trigonometric function as a variable, and solve for it.
   (A) Consider using algebraic manipulation such as factoring, combining or separating fractions, and so on.
   (B) Consider using identities.
2. After solving for a trigonometric function, solve for the variable.

In *solving a trigonometric equation using a graphing utility approach* you can solve a larger variety of problems than with the algebraic approach. The solutions are generally approximations (to whatever decimal accuracy desired).

# CHAPTER 7 Review Exercises

*Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems, except verifications, are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.*

*Verify each identity in Problems 1–4.*

**1.** $\tan x + \cot x = \sec x \csc x$

**2.** $\sec^4 x - 2\sec^2 x \tan^2 x + \tan^4 x = 1$

**3.** $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2\sec^2 x$

**4.** $\cos\left(x - \dfrac{3\pi}{2}\right) = -\sin x$

**5.** Write as a sum: $\sin 5\alpha \cos 3\alpha$.

**6.** Write as a product: $\cos 7x - \cos 5x$.

**7.** Simplify: $\sin\left(x + \dfrac{9\pi}{2}\right)$.

*Solve Problems 8 and 9 exactly (θ in degrees, x real).*

**8.** $\sqrt{2}\cos\theta + 1 = 0$, all $\theta$

**9.** $\sin x \tan x - \sin x = 0$, all real $x$

*Solve Problems 10–13 to four decimal places (θ in degrees and x real).*

**10.** $\sin x = 0.7088$, all real $x$

**11.** $\cos\theta = 0.2557$, all $\theta$

**12.** $\cot x = -0.1692$, $-\pi/2 < x < \pi/2$

**13.** $3\tan(11 - 3x) = 23.46$, $-\pi/2 < 11 - 3x < \pi/2$

**14.** Use a graphing utility to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of $x$ for which both sides are defined but are not equal.
(A) $(\sin x + \cos x)^2 = 1 - 2\sin x\cos x$
(B) $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

*Verify each identity in Problems 15–23.*

**15.** $\dfrac{1 - 2\cos x - 3\cos^2 x}{\sin^2 x} = \dfrac{1 - 3\cos x}{1 - \cos x}$

**16.** $(1 - \cos x)(\csc x + \cot x) = \sin x$

**17.** $\dfrac{1 + \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$

**18.** $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

**19.** $\cot\dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$

**20.** $\cot x - \tan x = \dfrac{4\cos^2 x - 2}{\sin 2x}$

**21.** $\left(\dfrac{1 - \cot x}{\csc x}\right)^2 = 1 - \sin 2x$

**22.** $\tan m + \tan n = \dfrac{\sin(m + n)}{\cos m \cos n}$

**23.** $\tan(x + y) = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$

*Evaluate Problems 24 and 25 exactly using appropriate sum–product or product–sum identities.*

**24.** $\cos 195° \sin 75°$

**25.** $\cos 195° + \cos 105°$

*In Problems 26–29, is the equation an identity? Explain.*

**26.** $\cot^2 x = \csc^2 x + 1$

**27.** $\cos 3x = \cos x(\cos 2x - 2\sin^2 x)$

**28.** $\sin(x + 3\pi/2) = \cos x$

**29.** $\cos(x - 3\pi/2) = \sin x$

*Solve Problems 30–34 exactly (θ in degrees, x real).*

**30.** $4\sin^2 x - 3 = 0$, $0 \le x < 2\pi$

**31.** $2\sin^2\theta + \cos\theta = 1$, $0° \le \theta \le 180°$

**32.** $2\sin^2 x - \sin x = 0$, all real $x$

**33.** $\sin 2x = \sqrt{3}\sin x$, all real $x$

**34.** $2\sin^2\theta + 5\cos\theta + 1 = 0$, all $\theta$

*Solve Problems 35–37 to four significant digits (θ in degrees, x real).*

**35.** $\tan\theta = 0.2557$, all $\theta$

**36.** $\sin^2 x + 2 = 4\sin x$, all real $x$

**37.** $\tan^2 x = 2\tan x + 1$, $0 \le x < \pi$

*Solve Problems 38–41 to four decimal places.*

**38.** $3\sin 2x = 2x - 2.5$, all real $x$

**39.** $3\sin 2x > 2x - 2.5$, all real $x$

**40.** $2\sin^2 x - \cos 2x = 1 - x^2$, all real $x$

**41.** $2\sin^2 x - \cos 2x \le 1 - x^2$, all real $x$

**42.** Given the equation $\tan(x + y) = \tan x + \tan y$:
(A) Is $x = 0$ and $y = \pi/4$ a solution?
(B) Is the equation an identity or a conditional equation? Explain.

**43.** Explain the difference in evaluating $\sin^{-1} 0.3351$ and solving the equation $\sin x = 0.3351$.

**44.** Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of $x$ for which both sides are defined but are not equal.
(A) $\dfrac{\tan x}{\sin x + 2\tan x} = \dfrac{1}{\cos x - 2}$
(B) $\dfrac{\tan x}{\sin x - 2\tan x} = \dfrac{1}{\cos x - 2}$

**45.** Use a sum or difference identity to convert $y = \cos(x - \pi/3)$ to a form involving $\sin x$ and/or $\cos x$. Enter the original equation in a graphing calculator as $y_1$, the converted form as $y_2$, and graph $y_1$ and $y_2$ in the same viewing window. Use TRACE to compare the two graphs.

**46.** (A) Solve $\tan(x/2) = 2\sin x$ exactly, $0 \le x < 2\pi$, using algebraic methods.
(B) Solve, $(x/2) = 2\sin x$, $0 \le x < 2\pi$, to four decimal places using a graphing calculator.

**47.** Solve $3\cos(x - 1) = 2 - x^2$ for all real $x$, to three decimal places using a graphing calculator.

*Solve Problems 48–50 exactly without the use of a calculator.*

**48.** Given $\tan x = -\frac{3}{4}$, $\pi/2 \le x \le \pi$, find
(A) $\sin(x/2)$ (B) $\cos 2x$

**49.** $\sin[2\tan^{-1}(-\frac{3}{4})]$

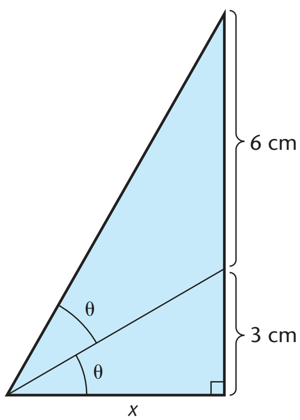
**50.** $\sin[\sin^{-1}(\frac{3}{5}) + \cos^{-1}(\frac{4}{5})]$

**51.** (A) Solve $\cos^2 2x = \cos 2x + \sin^2 2x$, $0 \le x < \pi$, exactly using algebraic methods.
(B) Solve $\cos^2 2x = \cos 2x + \sin^2 2x$, $0 \le x < \pi$, to four decimal places using a graphing utility.

**52.** We are interested in the zeros of $f(x) = \sin\dfrac{1}{x - 1}$ for $x > 0$.
(A) Explore the graph of $f$ over different intervals $[a, b]$ for various values of $a$ and $b$, $0 < a < b$. Does the function $f$ have a smallest zero? If so, what is it (to four decimal places)? Does the function have a largest zero? If so, what is it (to four decimal places)?
(B) Explain what happens to the graph as $x$ increases without bound. Does the graph have an asymptote? If so, what is its equation?
(C) Explore the graph of $f$ over smaller and smaller intervals containing $x = 1$. How many zeros exist on any interval containing $x = 1$? Is $x = 1$ a zero? Explain.

## APPLICATIONS

**53. INDIRECT MEASUREMENT** Find the exact value of $x$ in the figure, then find $x$ and $\theta$ to three decimal places. [*Hint:* Use a suitable identity involving $\tan 2\theta$.]




**54. ELECTRIC CURRENT** An alternating current generator produces a current given by the equation

$$I = 50\sin 120\pi(t - 0.001)$$

where $t$ is time in seconds and $I$ is current in amperes. Find the smallest positive $t$, to three significant digits, such that $I = 40$ amperes.

**55. MUSIC—BEAT FREQUENCIES** The equations

$$y = 0.6\cos 184\pi t \text{ and } y = -0.6\cos 208\pi t$$

model sound waves with frequencies 92 and 104 hertz, respectively. If both sounds are emitted simultaneously, a beat frequency results.
(A) Show that

$$0.6\cos 184\pi t - 0.6\cos 208\pi t = 1.2\sin 12\pi t \sin 196\pi t$$

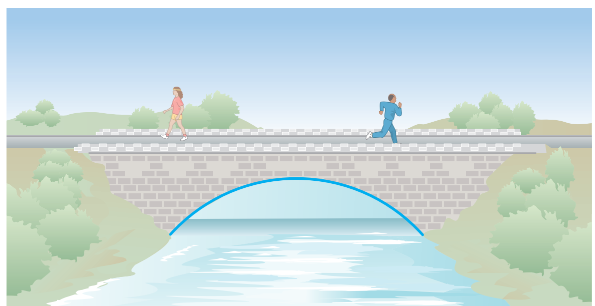
(B) Graph each of the following equations in a different viewing window for $0 \le t \le 0.2$.

$$y = 0.6\cos 184\pi t$$
$$y = -0.6\cos 208\pi t$$
$$y = 0.6\cos 184\pi t - 0.6\cos 208\pi t$$
$$y = 1.2\sin 12\pi t \sin 196\pi t$$

★**56. ENGINEERING** The circular arch of a bridge has an arc length of 36 feet and spans a 32-foot canal (see the figure). Determine the height of the circular arch above the water at the center of the bridge, and the radius of the circular arch, both to three decimal places. Start by drawing auxiliary lines in the figure, labeling appropriate parts, then explain how the trigonometric equation

$$\sin\theta = \frac{8}{9}\theta$$

is related to the problem. After solving the trigonometric equation for $\theta$, the radius is easy to find and the height of the arch above the water can be found with a little ingenuity.



## CHAPTER 7

### ››› GROUP ACTIVITY From $M \sin Bt + N \cos Bt$ to $A \sin (Bt + C)$— A Harmonic Analysis Tool

In solving certain kinds of more advanced applied mathematical problems—problems dealing with electrical circuits, spring-mass systems, heat flow, and so on—the solution process leads naturally to a function of the form

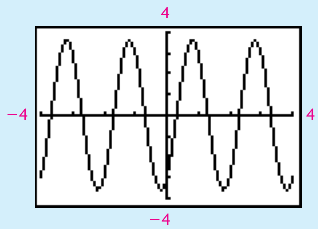
$$y = M \sin Bt + N \cos Bt \tag{1}$$

**(A)** *Graphing Utility Exploration.* Use a graphing utility to explore the nature of the graph of equation (1) for various values of $M$, $N$, and $B$. Does the graph appear to be simple harmonic; that is, does it appear to be a graph of an equation of the form $y = A \sin (Bt + C)$?

The graph of $y = 2 \sin (\pi t) - 3 \cos (\pi t)$ which is typical of the various graphs from equation (1), is shown in Figure 1. It turns out that the graph in Figure 1 can also be obtained from an equation of the form

$$y = A \sin (Bt + C) \tag{2}$$

for suitable values of $A$, $B$, and $C$.




› **Figure 1**
$y = 2 \sin (\pi t) - 3 \cos (\pi t)$.

The problem now is: given $M$, $N$, and $B$ in equation (1), find $A$, $B$, and $C$ in equation (2) so that equation (2) produces the same graph as equation (1). The form of equation (2) is often preferred over (1), because from (2) you can easily read amplitude, period, and phase shift and recognize a phenomenon as simple harmonic.

The process of finding $A$, $B$, and $C$, given $M$, $N$, and $B$, requires a little ingenuity and the use of the sum identity

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \tag{3}$$

How do we proceed? We start by trying to get the right side of equation (1) to look like the right side of identity (3). Then we use equation (3), from right to left, to obtain equation (2).

**(B)** *Establishing a Transformation Identity.* Show that

$$y = M \sin Bt + N \cos Bt = \sqrt{M^2 + N^2} \sin (Bt + C) \tag{4}$$

where $C$ is any angle (in radians if $t$ is real) having $P = (M, N)$ on its terminal side. [*Hint:* A first step is the following:

$$M \sin Bt + N \cos Bt = \frac{\sqrt{M^2 + N^2}}{\sqrt{M^2 + N^2}} (M \sin Bt + N \cos Bt)]$$

**(C)** *Use of Transformation Identity.* Use equation (4) to transform

$$y_1 = -4 \sin (t/2) + 3 \cos (t/2)$$

into the form $y_2 = A \sin (Bt + C)$, where $C$ is chosen so that $|C|$ is minimum. Compute $C$ to three decimal places. From the new equation, determine the amplitude, period, and phase shift.

**(D)** *Graphing Calculator Visualization and Verification.* Graph $y_1$ and $y_2$ from part C in the same viewing window.

**(E)** *Physics Application.* A weight suspended from a spring, with spring constant 64, is pulled 4 centimeters below its equilibrium position and is then given a downward thrust to produce an initial downward velocity of 24 centimeters per second. In more advanced mathematics (differential equations) the equation of motion (neglecting air resistance and friction) is found to be given approximately by

$$y_1 = -3 \sin 8t - 4 \cos 8t$$

where $y_1$ is the coordinate of the bottom of the weight in Figure 2 at time $t$ ($y$ is in centimeters and $t$ is in seconds). Transform the equation into the form

$$y_2 = A \sin (Bt + C)$$

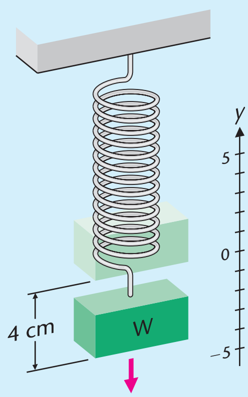



**Figure 2**
Spring-mass system.

and indicate the amplitude, period, and phase shift of the motion. Choose the least positive $C$ and keep $A$ positive.

**(F)** *Graphing Calculator Visualization and Verification.* Graph $y_1$ and $y_2$ from part E in the same viewing window of a graphing utility, $0 \le t \le 6$. How many times will the bottom of the weight pass $y = 2$ in the first 6 seconds?

**(G)** *Solving a Trigonometric Equation.* How long, to three decimal places, will it take the bottom of the weight to reach $y = 2$ for the first time?