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CALCULUS 1

FUNCTIONS, LIMITS AND DERIVATIVES FOR FIRST YEAR CALCULUS STUDENTS

FUNCTIONS

DEFINITIONS

• **FUNCTIONS.** A function is a correspondence that assigns one value (output) to each member of a given set. The given set of inputs is called the **domain**. The set of outputs is called the **range**. One-variable calculus deals with real-valued functions whose domain is a set of real numbers. If a domain is not specified, it is assumed to include all inputs for which there is a real number output.

• **NOTATION.** If a function is named f, then $f(x)$ denotes its value at x, or "f evaluated at x." If a function gives a quantity y in terms of a variable quantity x, then x is called the **independent variable** and y the **dependent variable**. Given a function by an equation such as $y = x^2$, one may think of y as a shorthand for the function's expression. The notation $x \mapsto x^2$ ("x maps to x^2") is another way to refer to the function. The *expression* $f(x)$ for a function at an arbitrary input x often stands in for the function itself.

• **GRAPHS.** The graph of a function f is the set of ordered pairs $(x, f(x))$, presented visually with a Cartesian coordinate system. The **vertical line test** states that a curve is the graph of a function if every vertical line meets the curve at most once. An equation $y=f(x)$ often refers to the set of points (x,y) satisfying the equation, in this case the graph of the function f. The **zeros** of a function are the inputs x for which $f(x)=0$, and they give the ***x*-intercepts** of the graph.

• **EVEN AND ODD.** A function f is
even if $f(-x) = f(x)$, e.g., x^2;
odd if $f(-x) = -f(x)$, e.g., x^3. Most are neither.

NUMBERS

• **RATIONAL NUMBERS.** A rational number is a ratio p/q of integers p and q, with $q \neq 0$. There are infinitely many ways to represent a given rational number, but there is a unique 'lowest-terms' representative. The set of all rational numbers forms a closed system under the usual arithmetic operations.

• **REAL NUMBERS.** In this chart, **R** denotes the set of real numbers. Real numbers may be thought of as the numbers representable by infinite decimal expansions. Rational numbers terminate in all zeros or have a repeating segment of digits. Real numbers that are not rational are called **irrational**. E.g., π, the ratio of circumference to diameter of a circle, is irrational; it may be *approximated* by rational numbers, e.g., 22/7 and 3.1416.

• **MACHINE NUMBERS.** A calculator or computer represents real numbers approximately using a fixed number of digits, usually between 8 and 16. Machine calculations are therefore usually not exact. This can cause anomalies in plots. The **precision** of a numerical result is the number of correct digits. (Count digits after appropriate rounding: 2.512 for 2.4833 has two correct digits.) The **accuracy** refers to the number of correct digits after the decimal point.

• **INTERVALS.** If $a < b$, the **open interval** (a,b) is the set of real numbers x such that $a < x < b$. The **closed interval** $[a,b]$ is the set of x such that $a \le x \le b$. The notation $(-\infty, a)$ denotes the 'half-line' consisting of all real numbers x such that $x < a$ (or $-\infty < x < a$). Likewise, there are intervals of the form $(-\infty, a]$, (a, ∞), and $[a, \infty)$.
The symbol ∞ is not to be thought of as a number, just a convenient symbol in these and other notations.
The whole real line is an interval, $\mathbf{R} = (-\infty, \infty)$.

NOTE TO STUDENT: Due to its condensed format, use this ***Quick Study®*** reference chart as a Calculus guide, but not as a replacement for assigned class work.

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NEW FROM OLD

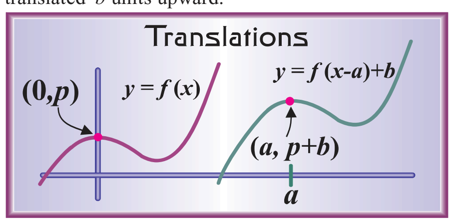
• **ARITHMETIC.** The **scalar multiple** of a function f by a constant c is given by $(cf)(x) = cf(x)$. The sum $f+g$, product fg, and quotient f/g of functions f and g are defined by:

$$(f+g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x)g(x),$$
$$(f/g)(x) = f(x)/g(x).$$

In each case, the domain of the new function is the intersection of the domains of f and g, with the zeros of g excluded for the quotient.

• **COMPOSITION.** If f and g are functions, "f composed with g" is the function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ with domain (strictly speaking) the set of x in the domain of g for which $g(x)$ is in the domain of f. E.g., $\sqrt{1-x^2}$ is the square root composed with $x \mapsto 1-x^2$, with domain $[-1,1]$.

• **TRANSLATIONS.** The graph of $x \mapsto (x-a)$ is the graph of f translated by a units to the right; e.g., $(a, f(0))$ would be on the graph. The graph of $x \mapsto f(x) + b$ is the graph of f translated b units upward.



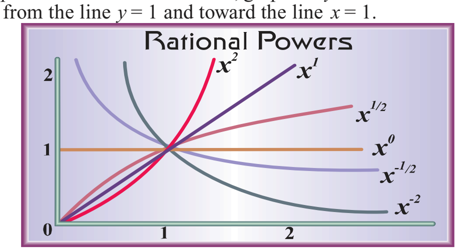
• **INVERSES.** An **inverse** of a function f is a function g such that $g(f(x)) = x$ for all x in the domain of f.
A function f has an inverse if and only if it is **one-to-one**: for each of its values y there is only one corresponding input; or, $f(x) = y$ has only one solution; or, any horizontal line meets the graph of f at most once. E.g., x^3 is one-to-one, x^2 is not. Strictly increasing or decreasing functions are one-to-one.
There can be only one inverse defined on the range of f, denoted $g = f^{-1}$. For any y in the range of f,
$f^{-1}(y)$ is the x that solves $f(x) = y$.
If the axes have the same scale, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

• **IMPLICIT FUNCTIONS.** A relation $F(x,y) = c$ often admits y as a function of x, in one or more ways. E.g., $x^2 + y^2 = 4$ admits $y = \sqrt{4-x^2}$. Such functions are said to be implicitly defined by the relation. Graphically, the relation gives a curve, and a piece of the curve satisfying the vertical line test is the graph of an implicit function. Often, there is no expression for an implicit function in terms of elementary functions. E.g., $x^2\,2^y + y^2\,2^x = 4$ admits $y = f(x)$ with $f(0) = 2$ and $f(2) = 0$, but there is no formula for $f(x)$.

• **RATIONAL POWERS.** These have the form

$$f(x) = x^{\frac{m}{n}} \equiv (x^m)^{\frac{1}{n}} \equiv (x^{\frac{1}{n}})^m$$

where it is assumed m and n are integers, $n > 0$, and $|m|/n$ is in lowest terms. If $m < 0$ then $x^m = 1/x^{|m|}$. The domain of $x^{m/n}$ is the same as that of the nth root function, excluding 0 if $m < 0$. For $x > 0$, as p decreases in absolute value, graphs of $y = x^p$ move toward the line $y = x^0 = 1$; as p increases in absolute value, graphs of $y = x^p$ move away from the line $y = 1$ and toward the line $x = 1$.



"Functions" *continued on next page...*

ELEMENTARY ALGEBRAIC FUNCTIONS

• **CONSTANT AND IDENTITY.**
A constant function has only one output:
$$f(x) = c.$$
The identity function is
$$x \mapsto x, \text{ or } f(x) = x.$$

• **ABSOLUTE VALUE.** $|x| = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \ge 0 \end{cases}$

The above is an example of a **piecewise definition**.

For any x, $\sqrt{x^2} = |x|$.

• **LINEAR FUNCTIONS.** For a linear function, the difference of two outputs is proportional to the difference of inputs. The proportionality constant, i.e., the ratio of output difference to input difference

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

is called the **slope**. The slope is also the change in the function per unit increase in the independent variable. The linear function

$$f(x) = mx + b$$

has slope m and ***y*-intercept** $f(0) = b$, the graph being a straight line. The slope has units the ratio of the units of the axes. (E.g., in a distance vs. time graph, slopes are velocities.) The linear function with value y_0 at x_0 and slope m is

$$f(x) = y_0 + m(x - x_0).$$

• **QUADRATICS.** These have the form
$$f(x) = ax^2 + bx + c \quad (a \neq 0).$$
The **normal form** is $f(x) = a(x-h)^2 + k$.
One has $h = -b/(2a)$ and $k = f(h)$. The graph is a parabola with **vertex** (h,k), opening up or down accordingly as $a > 0$ or $a < 0$, and symmetric about the vertical line through vertex. A quadratic has two, one, or no zeros accordingly as the **discriminant** $b^2 - 4ac$ is positive, zero, or negative. Zeros are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and are graphically located symmetrically on either side of the vertex.

• **POLYNOMIALS.** These have the form
$$p(x) = ax^n + bx^{n-1} + \cdots + dx + e.$$
Assuming $a \neq 0$, this has **degree** n, **leading coefficient** a, and **constant term** $e = p(0)$. A polynomial of degree n has at most n zeros. If x_0 is a zero of $p(x)$, then $x - x_0$ is a factor of $p(x)$:
$$p(x) = (x - x_0)\, q(x)$$
for some degree n-1 polynomial $q(x)$. A polynomial graph is smooth and goes to $\pm\infty$ when $|x|$ is large.

• **RATIONAL FUNCTIONS.** These have the form
$$f(x) = \frac{p(x)}{q(x)}$$
where $p(x)$ and $q(x)$ are polynomials. The domain excludes the zeros of q. The zeros of f are the zeros of p that are not zeros of q. The graph of a rational function may have vertical asymptotes and removable discontinuities, and is like that of some polynomial (perhaps constant) when $|x|$ is large.

• ***n*th ROOTS.** These have the form $y = x^{\frac{1}{n}} \equiv \sqrt[n]{x}$
for some integer $n > 1$. If n is even, the domain is $[0, \infty)$ and y is the unique nonnegative number such that $y^n = x$. If n is odd, the domain is **R** and y is the unique real number such that $y^n = x$. An nth root function is always increasing, the graph being vertical at the origin.

• **ALGEBRAIC VS. TRANSCENDENTAL.** An algebraic function $y = f(x)$ is one that satisfies a two-variable polynomial equation $P(x,y) = 0$. The functions above are algebraic. E.g., $y = |x|$ satisfies $x^2 - y^2 = 0$. Sums, products, quotients, powers, and roots of algebraic functions are algebraic. Functions that are not algebraic (e.g., exponentials, logarithms, and trig functions) are called transcendental.

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FUNCTIONS CONTINUED

EXPONENTIALS & LOGARITHMS

- **Pure exponentials.** The pure exponential function with **base** a ($a > 0$, $a \neq 1$) is

$$f(x) = a^x.$$

The domain is **R** and the range is $(0, \infty)$. The y - intercept is $a^0 = 1$. If $a < 1$ the function is decreasing; if $a > 1$ it is increasing. It changes by the factor $a^{\Delta x}$ over any interval of length Δx. Exponentials turn *addition into multiplication*:

$$a^0 = 1 \qquad a^{x+y} = a^x a^y \qquad a^{mx} = (a^x)^m$$
$$a^{x-y} = a^x / a^y \qquad a^{-x} = 1/a^x$$

- **Logarithms.** The logarithm with base a is the inverse of the base a exponential:

$$\log_a x = \text{"the power of } a \text{ that yields } x\text{"}.$$

Equivalently, $x = a^{\log_a x}$ or $\log_a a^y = y$.
The domain of $\log_a$ is $(0,\infty)$ and the range is **R**. If $a>1$ then $\log_a x$ is negative for $0 < x < 1$, positive for $x>1$, and always increasing. The **common logarithm** is $\log_{10}$. Examples:

$$\log_a a = 1 \qquad \log_2 32 = 5, \qquad \log_{10}(1/10) = -1$$

Logarithms turn *multiplication into addition*:

$$\log_a 1 = 0 \quad \log_a xy = \log_a x + \log_a y \quad \log_a x^m = m\log_a x$$
$$\log_a (x/y) = \log_a x - \log_a y \qquad \log_a(1/x) = -\log_a x$$

The third identity holds for any real number m. For a **change of base**, one has

$$\log_b x = \log_a x \cdot \log_b a = \frac{\log_a x}{\log_a b}$$

- **Natural exponential and logarithm.** The natural exponential function is the pure exponential whose tangent line at the point (0,1) on its graph has slope 1. Its base is an irrational number

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718.$$

The **natural logarithm** is $\ln \equiv \log_e$, the inverse to $x \mapsto e^x$:

$$\ln x = y \quad \text{means } x = e^y.$$

There are identities

$$\ln e^x = x, \qquad e^{\ln x} = x, \qquad \ln e = 1,$$

and ln has the properties of a logarithm.
E.g., $\ln(1/x) = -\ln x$. Special values are:

$$\ln 1 = 0, \qquad \ln 2 \approx 0.6931, \qquad \ln 10 \approx 2.303.$$

Any exponential can be written $a^x = e^{(\ln a)x}$.

Any logarithm can be written $\log_a x = \dfrac{\ln x}{\ln a}$.

- **General exponential functions.** These have the form $f(x) = P_0 a^x$ and have the property that the ratio of two outputs depends only on the difference of inputs. The ratio of outputs for a unit change in input is the **base** a. The y -intercept is $f(0) = P_0$.

- **Exponential growth.** A quantity P (e.g., invested money) that increases by a factor $a = e^r > 1$ over each unit of time is described by

$$P = P_0\, a^t = P_0 e^{rt}.$$

Over an interval Δt the factor is $a^{\Delta t}$. E.g., if P increases 4% each half year, then $a^{1/2} = 1.04$, and

$$P = P_0(1.04)^{2t} \approx P_0 e^{0.078t} \qquad (t \text{ in yrs}).$$

The **doubling time** D is the time interval over which the quantity doubles:

$$a^D = e^{rD} = 2, \qquad D = \frac{\ln 2}{\ln a} = \frac{\ln 2}{r}.$$

If the doubling time is D, then $P = P_0\, 2^{t/D}$.

- **Continuous compounding** at the annual percentage rate r x 100% yields the annual growth factor

$$a = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

- **Exponential decay.** A quantity Q (e.g., of radioactive material) that decreases to a proportion $b = e^{-k} < 1$ over each unit of time is described by

$$Q = Q_0\, b^t = Q_0\, e^{-kt}.$$

Over an interval Δt the proportion is $b^{\Delta t}$. E.g., if Q decreases 10% every 12 hours, then $b^{12} = 0.90$, and

$$Q = Q_0(0.90)^{t/12} \approx Q_0 e^{-.0088t} \quad (t \text{ in hrs}).$$

The **half-life** H is the time interval over which the quantity decreases by the factor one-half:

$$b^H = e^{-kH} = \frac{1}{2}, \qquad H = \frac{\ln 2}{\ln b} = \frac{\ln 2}{k}.$$

If the half-life is H, then $Q = Q_0(1/2)^{t/H}$.

- **Irrational powers** These may be defined by

$$f(x) = x^p = e^{p\ln x} \qquad (x > 0).$$

- **Hyperbolic functions.** The **hyperbolic cosine** is $\cosh x = \dfrac{e^x + e^{-x}}{2}$. It has domain **R**, range $[1,\infty)$, and is even. On the restricted domain $[0,\infty)$, it has inverse

$$\operatorname{arccosh} x \equiv \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right).$$

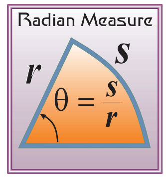
The **hyperbolic sine** is $\sinh x = \dfrac{e^x - e^{-x}}{2}$.
It has domain **R**, range **R**, and is odd.
Always strictly increasing, it has inverse

$$\operatorname{arcsinh} x \equiv \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$$

The basic identity is $\cosh^2 x - \sinh^2 x = 1$.

TRIGONOMETRIC FUNCTIONS

- **Radians.** The radian measure of an angle θ is the ratio of length s to radius r of a corresponding circular arc: $\theta = \frac{s}{r}$.

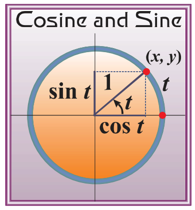


$$2\pi \text{ radians} = 360^\circ$$
$$1^\circ = \frac{\pi}{180}\text{radians}$$

In calculus, it is normally assumed (and necessary for standard derivative formulas) that arguments to trig functions are in radians.

- **Cosine, sine, tangent.** Consider a real number t as the radian measure of an angle: the distance measured counter-clockwise along the circumference of the *unit* circle from the point (1,0) to a terminal point (x,y). Then

$$\cos t = x; \quad \sin t = y;$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{y}{x}.$$



Cosine and sine have domain **R** and range [-1,1]. The domain of the tangent excludes $\pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \ldots$, and its range is **R**. The cosine is even, the sine and tangent are odd.

- **Secant, cosecant, cotangent.**

$$\sec t = \frac{1}{\cos t};\ \csc t = \frac{1}{\sin t};\ \cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

- **Special values.**

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\cos t$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\sin t$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
$\tan t$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0

- **Identities.**

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Other identities are obtained from the above. E.g.,

$$\sin(t - \pi/2) = -\cos t$$
$$\cos(2t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

For the last, divide sum identities by $\cos a \ \cos b$.

- **Amplitudes, periods, & phases.** If

$$f(t) = A\sin(\omega t + \phi) + k$$

with $A > 0$, $\omega > 0$, and $-\pi < \phi \le \pi$, then the **amplitude** is A, the **average value** is k, the **period** is $2\pi/\omega$, the **frequency** $\left(1/\text{period}\right)$ is $\omega/2\pi$, the **angular frequency** is ω, and the **phase shift** (relative to $A\sin\,\omega t$) is ϕ.

- **Inverse trig functions.**
The **arccosine** is inverse to cosine on $[0,\pi]$:

arccos x = "angle in $[0,\pi]$ whose cosine is x"

It has domain [-1,1] and range $[0,\pi]$.

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}, \quad \arccos\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

The **arcsine** is inverse to sine on : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

arcsin x = "angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x."

It has domain [-1,1], range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and is odd.

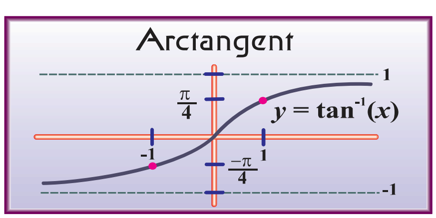
The **arctangent** is inverse to tangent on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

arctan x = "angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with tangent x."

It has domain **R**, range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and is odd.

$$\arctan\sqrt{3} = \frac{\pi}{3}, \quad \arctan(-1) = -\frac{\pi}{4}$$

The notation $\cos^{-1} x$ for arccos x is not to be confused with $1/\cos x$; likewise $\sin^{-1} x$ and $\tan^{-1} x$.



LIMITS

DEFINITIONS

- **Limit.** Intuitively, the limit of $f(x)$ as x approaches a is the number that $f(x)$ gets close to when x gets close to a. Precisely, a number L is the limit, written

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad f(x) \to L \ \text{ as } \ x \to a$$

if every $\varepsilon > 0$ admits a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \ \text{ when } \ 0 < |x - a| < \delta.$$

It is assumed that $f(x)$ is defined for all x in some open interval containing a, except perhaps $x = a$. If a limit exists, there is only one. The limit statement says nothing about the value of f at $x = a$.

- **Zooming formulation.** If the plot range for f is held fixed with L in the middle, and the plot domain is narrowed through intervals centered at $x = a$, the graph of f eventually lies completely within the fixed plot range, except perhaps at $x = a$. (Compare with Zooming Formulation under **Continuity**, next page.)

- **One-sided limits.** The left-hand limit is equal to L, written

$$\lim_{x\to a^-} f(x) = L \ \text{ or } \ f(a^-) = L,$$

if every $\varepsilon > 0$ admits a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \ \text{ when } \ a - \delta < x < a.$$

The right-hand limit is defined similarly, the last condition being $a < x < a + \delta$. E.g.,

$$\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

A limit exists if and only if the left and right-hand limits exist and are equal.

- **Infinite limits.** One writes $\lim_{x\to a} f(x) = \infty$ if every $Y > 0$ admits a $\delta > 0$ such that

$$f(x) > Y \ \text{ when } \ 0 < |x - a| < \delta.$$

Likewise, there are one-sided limits to ∞, and limits to $-\infty$. E.g., $\lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty$.

- **Limits at infinity.** One writes

$$\lim_{x\to\infty} f(x) = L$$

if every $\varepsilon > 0$ admits an $X > 0$ such that

$$|f(x) - L| < \varepsilon \ \text{ when } \ x > X.$$

LIMIT THEOREMS

- **Note.** The following theorems have counterparts involving limits to infinity. Also, "for x near a" will mean "for all x in some open interval containing a, except perhaps $x = a$."

- **Arithmetic.** A limit of a sum is the sum of the individual limits, provided each individual limit exists. Likewise for a limit of a difference or a product. The limit of a quotient is the quotient of the individual limits, provided each limit exists and the limit of the denominator is nonzero. If c is a scalar, then

$$\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x).$$

- **Compositions.** If $\lim_{x\to a} g(x) = l$ and $g(x) \ne l$ for x near a (or if F is continuous at l), then

$$\lim_{x\to a} F(g(x)) = \lim_{y\to l} F(y)$$

provided the limit on the right exists. E.g.,

$$\lim_{x\to 1}(x^2+1)^n = \lim_{y\to 2} y^n = 2^n$$
$$\lim_{x\to 0}\frac{\sin 5x}{5x} = \lim_{y\to 0}\frac{\sin y}{y} = 1$$

- **Inequalities.** If $f(x) \le M$ for x near a, then $\lim_{x\to a} f(x) \le M$ if the limit exists. Likewise if $f(x) \ge m$.

- **Sandwich Theorem.** If $g(x) \le f(x) \le h(x)$ for x near a, and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.
Special case: if $|f(x)| \le h(x)$ for x near a, then

$$\lim_{x\to a} h(x) = 0 \ \text{ implies } \ \lim_{x\to a} f(x) = 0.$$

E.g., $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ (using $h(x) = |x|$).

- **L'Hôpital's Rule.** (Needs derivatives.) If

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x),$$

and if $f'(x)$ and $g'(x)$ are defined and $g'(x) \ne 0$ for x near a, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the latter limit exists (or is infinite). The rule also holds when the limits of f and g are $\pm\infty$.

QuickStudy

LIMIT FORMULAS

- **Polynomials and rational functions.**

If c is a constant, $\lim_{x\to a} c = c$.

If $p(x)$ is a polynomial, $\lim_{x\to a} p(x) = p(a)$.

Let $p(x)$ and $q(x)$ be polynomials.

If $q(a) \neq 0$, then $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$.

If $q(a) = 0$ and $p(a) \neq 0$, one-sided limits are $\pm\infty$.

E.g., for integer $n > 0$, $\lim_{x\to 0^+} \frac{1}{x^n} = \infty$;
$\lim_{x\to 0^-} \frac{1}{x^n} = -\infty$ (n odd); $\lim_{x\to 0^-} \frac{1}{x^n} = \infty$ (n even).

If $q(a) = 0$ and $p(a) = 0$, first cancel all common factors of $x - a$ from $p(x)$ and $q(x)$. E.g.,

$$\lim_{x\to 1^-} \frac{x^2+2x-3}{x^3-3x+2} = \lim_{x\to 1^-} \frac{x+3}{(x-1)(x+2)} = -\infty.$$

- **Rational functions at infinity.**

For integer $n > 0$, $\lim_{x\to\infty} x^n = \infty$;
$\lim_{x\to-\infty} x^n = -\infty$ (n odd); $\lim_{x\to-\infty} x^n = \infty$ (n even); $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$

$$\lim_{x\to\pm\infty} \frac{ax^n + bx^{n-1} + \ldots}{cx^m + dx^{m-1} + \ldots} = \frac{a}{c} \lim_{x\to\pm\infty} x^{n-m} \quad \text{(a, c non zero)}$$

- **Arbitrary powers.**

$\lim_{x\to a} x^p = a^p$ (when a^p is defined)

For $p > 0$, $\lim_{x\to\infty} x^p = \infty$ and $\lim_{x\to\infty} x^{-p} = 0$.

- **Limits for basic derivatives**

$\lim_{x\to a} \frac{x^m - a^m}{x-a} = ma^{m-1}$ (when a^{m-1} is defined)

$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ (a definition of e) $\quad \lim_{h\to 0} \frac{a^h - 1}{h} = \ln a$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

CONTINUITY

DEFINITIONS

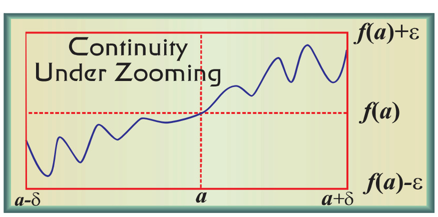
- **Continuity at a point.** A function f is **continuous at** a if a is in the domain of f and

$$\lim_{x\to a} f(x) = f(a).$$

Explicitly, f is defined on some open interval containing a, and every $\varepsilon > 0$ admits a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ when } |x - a| < \delta.$$

- **Zooming formulation.** If the plot range for f is held fixed with $f(a)$ in the middle, and the plot domain is narrowed through intervals centered at $x = a$, the graph of f eventually lies completely within the fixed plot range. This must hold for any such plot range.



- **One-sided continuity.** A function f is **continuous from the left at** a if a is in the domain of f and

$$\lim_{x\to a^-} f(x) = f(a).$$

A function f is **continuous from the right at** a if a is in the domain of f and

$$\lim_{x\to a^+} f(x) = f(a).$$

- **Global continuity.** We say a function is **continuous** if it is continuous on its domain, meaning continuous at every point in its domain, using one-sided continuity at endpoints of intervals. Caution: textbooks sometimes refer to some points not in the domain as points of discontinuity. Intuitively, a function is continuous on an interval if there are no breaks in its graph.

- **Uniform continuity.** A function f is **uniformly continuous** on its domain D if for every $\varepsilon > 0$ there is a $\delta > 0$ such that x,y in D and $|x-y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Uniform continuity implies continuity. A continuous function on a closed interval $[a,b]$ is uniformly continuous.

DERIVATIVES

DEFINITIONS

- **Derivative.** The derivative of f at a is the number

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists, in which case f is said to be **differentiable** at a. The derivative of f is the function f'. The derivative is also

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

by the limit theorem for compositions applied to

$$x \mapsto F(x-a), \text{ with } F(h) = \frac{f(a+h) - f(a)}{h}.$$

- **Zooming formulation.** If the plot domain for f is narrowed through intervals centered at $x = a$ while the ratio of the plot range to the plot domain is held fixed, the graph of f eventually appears linear (identical to the **tangent line** at $x = a$). If $f'(a) \neq 0$, the zoomed graph appears linear with no constraint on the plot ranges ('auto-scaling').

- **Notation.** The derivative function itself is denoted f' or $D(f)$. If $y = f(x)$, the following usually represent *expressions* for the derivative function:

$$y', \quad \frac{dy}{dx}, \quad D_x y, \quad f'(x), \quad \frac{d}{dx} f(x).$$

The second is the **Liebniz notation**. Notations for the derivative evaluated at $x = a$ are

$$f'(a), \quad D(f)(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad \left.\frac{d}{dx}\right|_{x=a} f(x).$$

- **Linearization.** The linearization, or **linear approximation**, of f at a is the linear function

$$x \mapsto f(a) + f'(a)(x-a).$$

Its graph is the **tangent line** to the graph of f at the point $(a,f(a))$. The derivative thus provides a 'linear model' of the function near $x = a$.

- **Differentials.** The differential of f at a is the expression

$$df(a) = f'(a)dx.$$

Applied to an increment Δx, it becomes $f'(a)\Delta x$. If $y = f(x)$, one writes $dy = f'(x)dx$.

- **Difference quotients.** The difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

approximates $f'(a)$ if h is small. It is the slope of the **secant line** through the points $(a,f(a))$ and $(a+h,f(a+h))$. The average of it and the 'backward quotient',

$$\frac{f(a) - f(a-h)}{h}$$

is the **symmetric quotient**

$$\frac{f(a+h) - f(a-h)}{2h}$$

usually a better approximation of $f'(a)$.

THEORY

- **Arithmetic.** Scalar multiples of a continuous function are continuous. Sums, differences, products, and quotients of continuous functions are continuous (*on their domains*).

- **Compositions.** A composition of continuous functions is continuous.

- **Elementary functions.** Polynomials, rational functions, root functions, exponentials and logarithms, and trigonometric and inverse trigonometric functions are continuous.

- **Intermediate value theorem.** If f is continuous on the closed interval $[a,b]$, then f achieves every value between $f(a)$ and $f(b)$: for every y between $f(a)$ and $f(b)$ there is at least one x in $[a,b]$ such that $f(x) = y$. The **zero theorem** states that if f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is an x in (a,b) such that $f(x) = 0$.

- **Bisection Method.** This a method of finding zeros based on the zero theorem.
  1. With f, a, b as in the zero theorem, the midpoint $x_1 = \frac{1}{2}(a+b)$ is an initial estimate of a zero.
  2. Assuming $f(x_1)$ is nonzero, there is a new interval $[a,x_1]$ or $[x_1,b]$ on which opposite signs are taken at the endpoints. It contains a zero, and its midpoint x_2 is a new estimate of a zero.
  3. Repeat step (2) with the new interval and x_2.
  4. The nth estimate x_n differs from a zero by no more than $(b-a)/2^n$.

- **Extreme value theorem.** If f is continuous on the closed interval $[a,b]$, then f achieves a minimum and a maximum on $[a,b]$: there are c and d in $[a,b]$ such that $f(c) \le f(x) \le f(d)$ for all x in $[a,b]$. The proofs of this and the intermediate value theorem use properties of the set of real numbers not covered in introductory calculus.

INTERPRETATIONS

- **Rate of change.** The derivative $f'(a)$ is the **instantaneous rate of change** of f with respect to x at $x = a$. It tells how fast f is increasing or decreasing as x increases through values near $x = a$. The **average rate of change** of f over an interval $[a,x]$ is $\frac{f(x) - f(a)}{x-a}$. As x nears a, these average rates approach $f'(a)$. The **units of the derivative** are the units of $f(x)$ divided by the units of x.

- **Tangent line.** The derivative $f'(a)$ is the slope of the tangent line to the graph of f at the point $(a,f(a))$. It is a limit of slopes of secant lines passing through that point.

- **Linear Approximation.** One can approximate values of f near a according to

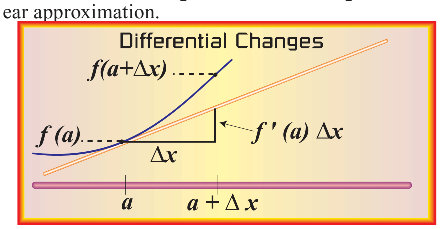
$$f(x) \approx f(a) + f'(a)(x-a).$$

E.g., since $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, $\sqrt{62} \approx \sqrt{64} + \frac{1}{2\sqrt{64}}(-2) = 7.875$.
The approximation is better the closer x is to a and the flatter the graph is near a.

- **Differential changes.** At a given input, the derivative is the factor by which small input changes are scaled to become approximate output changes. The differential change at a over an input increment Δx approximates the output change:

$$f'(a)\Delta x \approx f(a+\Delta x) - f(a).$$

The differential change is the exact change in the linear approximation.



- **Velocity.** Suppose $s(t)$ is the position at time t of an object moving along a straight line.
Its **average velocity** over a time interval t_0 to t_1 is

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Its **instantaneous velocity** at time t is $v(t) = s'(t)$. Its **speed** is $|v(t)|$. Its **acceleration** is $v'(t)$.

- **Interpreting a derivative value.** Suppose T is temperature (in °C) as a function of location x (in cm) along a line. The meaning of, say, $T'(8) = 0.31$ (°C/cm) is that, at the location $x = 8$, small shifts in the positive x direction yield small increases in temperature in a ratio of about 0.31 °C per cm shift. Small shifts in the negative direction yield like decreases in T.

APPLICATIONS

- **Linear approximations at 0.**
The following are commonly used linear approximations valid near $x = 0$.

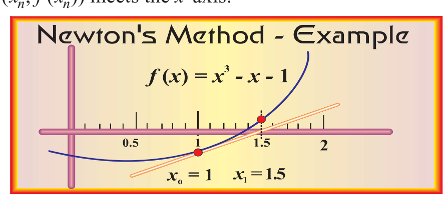
$\sin x \approx x \qquad \tan x \approx x \qquad e^x \approx 1+x$
$\ln(1+x) \approx x \qquad (1+x)^{1/2} \approx 1+x/2 \qquad 1/(1+x) \approx 1-x$

The error in each approximation is no more than $M|x|^2/2$, where M is any bound on $|f''(y)|$ for $|y| \le |x|$, f being the relevant function. E.g., $|\sin x - x| \le .005$ for $|x| \le 0.1$.

- **Newton's method.** To find an approximate root of $f(x) = 0$, select an appropriate starting point x_0, and evaluate

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

successively for $n = 0, 1, \ldots$, until the values do not change at the desired precision. The value on the right hand side in the above is where the tangent line at $(x_n, f(x_n))$ meets the x-axis.



- **Related rates.** Suppose two variables, each a function of 'time', are related by an equation. Differentiate both sides of the equation with respect to time to get a relation involving the time derivatives—the rates—and the original variables. With sufficient data for the variables and one of the rates, the derivative relation can be solved for the other rate.

"Derivatives" continued on next page

ANALYSIS

DIFFERENTIATION RULES

- **General notes.** In the following, assume *f* and *g* are differentiable. Each rule should be viewed as saying that the function to be differentiated is differentiable on its domain and that the derivative is as given. For each, there is also a functional form, e.g., $(cf)' = cf'$, and a Liebniz form, e.g., $\frac{d}{dx}(cu) = c\frac{du}{dx}$.
- **Sum.** $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$
- **Scalar multiple.** $\frac{d}{dx}[cf(x)] = cf'(x)$
- **Product.** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$
- **Quotient.** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$
- **The Chain Rule (for compositions).**

$$\frac{d}{dx}[(f \circ g)(x)] \equiv \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

This says that a small change in input to the composition is scaled by $g'(x)$, then by $f'(g(x))$. In Liebniz notation, if $z=f(y)$ and $y=g(x)$, and we thereby view z a function of x, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}, \quad \frac{dz}{dy} \text{ being evaluated at } y=g(x).$$

In D notation, $D(f \circ g) = [D(f) \circ g]D(g)$.

- **Inverse functions.** If *f* is the inverse of a function *g* (and g' is continuous and nonzero), then

$$f'(x) = \frac{1}{g'(f(x))}.$$

To get a specific formula directly, start with $y=f(x)$; rewrite it $g(y)=x$; differentiate with respect to x to get $g'(y)y'=1$; write this $y'=1/g'(y)$ and put $g'(y)$ in terms of x, using the relations $y=f(x)$ and $g(y)=x$.

E.g., $y=\ln x;\ e^y=x;\ e^y y'=1;\ y'=1/e^y=1/x$.

- **Implicit functions.** The derivative of a function defined implicitly by a relation $F(x,y)=c$ may be found by differentiating the relation with respect to x while treating y as a function of x wherever it appears in the relation; and then solving for y' in terms of x and y. The result is the same as obtained from the formal expression

$$y' = -\frac{\frac{d}{dx}F(x,y)}{\frac{d}{dy}F(x,y)},$$

where y is treated as a constant in the numerator, x as a constant in the denominator.

DERIVATIVE FORMULAS

- **Constants.** For any constant C, $\frac{d}{dx}C = 0$.
- **Reciprocal function.** $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$

The chain rule gives $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{1}{f(x)^2}f'(x)$

- **Square root.** $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
- **Powers.** For any real value of n, $\frac{d}{dx}x^n = nx^{n-1}$, valid where x^{n-1} is defined. The chain rule gives

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x).$$

- **Exponentials.** An exponential function has derivative proportional to itself, the proportionality factor being the natural logarithm of the base:

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}a^x = (\ln a)a^x$$

The chain rule gives $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$

- **Logarithms.**

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad \frac{d}{dx}\log_a|x| = \frac{1}{(\ln a)x}$$

Same rules hold without absolute value, but the domain is restricted to $(0,\infty)$. The chain rule gives

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}.$$

- **Hyperbolic functions.**

$$\sinh' x = \cosh x \qquad \cosh' x = \sinh x$$

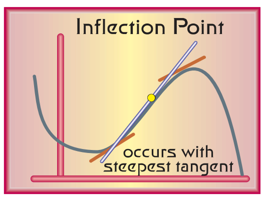
$$\text{arcsinh}' x = \frac{1}{\sqrt{1+x^2}} \qquad \text{arccosh}' x = \frac{1}{\sqrt{x^2-1}}$$

- **Trig functions.**

$$\sin' x = \cos x \qquad \cos' x = -\sin x$$
$$\tan' x = \sec^2 x \qquad \cot' x = -\csc^2 x$$
$$\sec' x = \sec x \tan x \qquad \csc' x = -\csc x \cot x$$
$$\arcsin' x = \frac{1}{\sqrt{1-x^2}} = -\arccos' x$$
$$\arctan' x = \frac{1}{1+x^2} = -\text{arccot}' x$$

LOCAL FEATURES OF FUNCTIONS

- **Neighborhoods.** In the following, "near" a point means in an open interval containing the point. Such an open interval is often called a neighborhood of the point.
- **Continuity.** If a function is differentiable at a point, then it is continuous there.
- **Critical points.** A point c is a critical point of f if f is defined near c and either $f'(c)=0$ or $f'(c)$ does not exist.
- **Local extrema.** A **local minimum** point of f is a point c with $f(x) \geq f(c)$ for x near c. A **local maximum** point of f is a point c with $f(x) \leq f(c)$ for x near c. If c is a local extremum point, then it is a critical point. (This follows from definitions.) **Relative extrema** are the same as local extrema.
- **First Derivative Test.** Suppose c is a critical point of f and f is continuous at c. If $f'(x)$ changes sign from *negative to positive* as x increases through c, then c is a local *minimum* point. If $f'(x)$ changes sign from *positive to negative* as x increases through c, then c is a local *maximum* point. If $f'(x)$ keeps the same sign, then c is not an extremum point.
- **Second Derivative Test.** Suppose f is differentiable near a critical point c. If $f''(c) > 0$, then c is a local minimum point. If $f''(c) < 0$, then c is a local maximum point.
- **Inflection points.** If the graph of f has a tangent line (possibly vertical) at c and $f'(x)$ changes sign as x increases through c, then c, or the graph point $(c,f(c))$, is called an inflection point. E.g., $x^{1/3}$ has a vertical tangent and inflection point at (0,0). An inflection point for f is an extremum for f'; the tangent line is locally steepest at such a point. The only possible inflection points are where $f''(x)=0$ or $f''(x)$ does not exist.

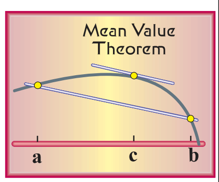


TRENDS & GLOBAL FEATURES

- **Mean Value Theorem (MVT).** If f is continuous on $[a,b]$ and differentiable on the open interval (a,b), then there is a point c in (a,b) with

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Graphically, some tangent line between a and b is parallel to the secant line through $(a,f(a))$ and $(b,f(b))$. The case with $f(a)=f(b)=0$, whence $f'(c)=0$, is **Rolle's Theorem.** The proof of the MVT relies on the Extreme Value Theorem.



- **Increasing and decreasing.** If $f'=0$ on an interval, then f is constant on that interval. If $f'>0$ on an interval, then f is strictly increasing on that interval. If $f'<0$ on an interval, then f is strictly decreasing on that interval. (These follow from MVT.)
- **Concavity.** A graph is said to be **concave up [down]** at a point c if the graph lies above [below] the tangent line near c, except at c. If $f''>0$ on an interval, then the graph of f is concave up on the interval (UP-POSITIVE); also f' is increasing, and the tangent lines are turning upward as x increases. If $f''<0$ on an interval, then the graph of f is concave down on the interval (DOWN-NEGATIVE); also f' is decreasing, and the tangent lines are turning downward as x increases.
- **Extrema on a closed interval.** The global, or absolute, maximum and minimum values of a continuous function on a closed interval $[a,b]$ (guaranteed to be achieved by the Extreme Value Theorem) can only occur at critical points or endpoints.

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APPLICATIONS

- **Optimization with constraint.** Here is an outline to approach optimization problems involving two variables that are somehow related.
  1. Visualize the problem and name the variables.
  2. Write down the **objective function**—the one to be optimized—as a function of two variables.
  3. Write down a **constraint equation** relating the variables.
  4. Use the constraint to rewrite the objective function in terms of one variable.
  5. Analyse the new function of one variable to find its optimal point(s), and the optimal value.

  E.g., to maximize the area of a rectangle with perimeter being p, we pose the problem as maximizing $A=lw$ subject to the constraint $2l+2w=p$. The constraint gives $w=p/2-l$, whence $A=l(p/2-l)$. The maximum occurs at $l=p/4$, with $A=(p/4)^2$. A verbal result is clearest: it's a square.

For geometric problems, **volume formulas** may be needed: cylinder: $\pi r^2 h$, cone: $\pi r^2 h/3$, sphere: $4\pi r^3/3$.

- **Cubics.** A cubic $p(x)=ax^3+bx^2+cx+d$ has exactly one inflection point: (h,k) where $h=-b/(3a)$ and $k=p(h)$. A **normal form** is

$$p(x) = a(x-h)^3 + m(x-h) + k$$

where $m=bh+c$ is the slope at the inflection point. If m and a have opposite signs, the horizontal line through the inflection point meets the graph at two points, each a distance $\sqrt{-m/a}$ from the inflection point, and local extrema occur at points $1/\sqrt{3} \approx 0.6$ times that distance.

INTEGRATION

INTERPRETATIONS

- **Area under a curve.** The integral of a nonnegative function over an interval gives the area under the graph of the function.
- **Average value.** The average value of f over an interval $[a,b]$ may be defined by

$$\text{average value} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Often a rough estimate of an integral can be made by estimating the average value (by inspection of the graph, say) and multiplying it by the length of the interval.

- **Accumulated change.** The integral of a rate of change gives the total change in the original quantity over the time interval. E.g., if $v(t)=s'(t)$ represents velocity, then $v(t)\Delta t$ is the approximate displacement occurring in the time increment t to $t+\Delta t$. Adding the displacements for all the time increments gives the approximate change in position over the entire time interval. In the limit of small time increments, one gets the integral $\int_a^b v(t)dt = s(b)-s(a)$, which is the total displacement.

FUNDAMENTAL THEOREM OF CALCULUS

- **Antiderivatives.** An antiderivative of a function f is a function F whose derivative is f: $F'(x)=f(x)$ for all x in some domain. Any two antiderivatives of a function on an interval differ by a constant. (This follows from MVT.) E.g., arctan x and -arctan$(1/x)$ are both antiderivatives of $1/(1+x^2)$ for $x>0$. (They differ by $\pi/2$.) An antiderivative is also called an **indefinite integral**, though the latter term often refers to the entire family of antiderivatives.
- **The Fundamental Theorem.** There are two parts:
  1) **Evaluating integrals.** If f is continuous on $[a,b]$, and F is any antiderivative of f on that interval, then

  $$\int_a^b f(x)dx = F(x)\Big|_a^b \equiv F(b)-F(a).$$

  2) **Constructing antiderivatives.** If f is continuous on $[a,b]$, then the function

  $$G(x) = \int_a^x f(w)\,dw$$

  is an antiderivative of f on (a,b): $G'(x)=f(x)$. (The one-sided derivatives of G agree with f at the endpoints.)
- **Differentiation of integrals.** To differentiate a function such as $x \mapsto \int_a^{x^2} f(w)dw$, view it as a composition $G(x^2)$, with G as above. The chain rule gives

$$\frac{d}{dx}G(x^2) = G'(x^2)\cdot 2x = 2xf(x^2).$$