Exercise 1:- Let E, F be Banach spaces and $(T_i)_{i\in I} \in \mathcal{L}(E,F)$ where I is not necessarily countable. Suppose that $\forall x \in E$, $\{T_i(x),\, i \in I\}$ is bounded in F. Then there exists $M \geq 0$ such that $\forall\, i \in I$, $\|T_i\|_{\mathcal{L}(E,F)} \leq M$.

Proof:

Let $E_n = \{x \in E : \|T_i x\| \leq n \quad \forall i \in I\}$,

which are closed sets and where $\sup\limits_{i\in I} \|T_i x\| < \infty \quad \forall x \in E$

then we have

$$E = \bigcup_{n=1}^{\infty} E_n .$$

By using "Bair Category Theorem", some E_n has nonempty interior

i.e. For some $n_0 \geq 1$, then $\text{Int}(E_{n_0}) \neq \phi$.

So, we can say $B(x_0, r) \subseteq E_N$.

Therefore, for an $z \in B(0,1)$

$$\| T_i(x_0 + r z)\| \leq n$$

which implies that $\|T_i\| \leq \frac{1}{r}\left(N + \|T_i x_0\|\right)$. #

Exercise 2: Let G be a Banach space and let B be a subset of G. Suppose that $\forall f \in G^*$ we have $f(B) = \{f(x), x \in B\}$ is bounded in $\mathbb{R}$. Then B is bounded.

Proof:

By using (Exercise 1) with $E = G^*$, $F = \mathbb{R}$ and $I = B$. For every $b \in B$, set

$$T_b(f) = \langle f, b \rangle = f(b), \quad f \in E = G^*,$$

where $f(B)$ is bounded, then

$$\sup_{b \in B} |T_b(f)| < \infty \quad \forall f \in E.$$

It follows from last Exersise that there exists a Constant $M \geq 0$ s.t

$$|T_b(f)| \leq M \quad \forall f \in G^*, \forall b \in B$$

Since $\|T_b(f)\| \leq \|b\| \, \|T\|$

then $\|b\| \leq \frac{M}{\|T\|} \quad \forall b \in B \Rightarrow B$ is bounded #