Royal Commission for Jubail and Yanbu

Jubail University College

Department of Mechanical Engineering

Numerical Methods

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Math 314

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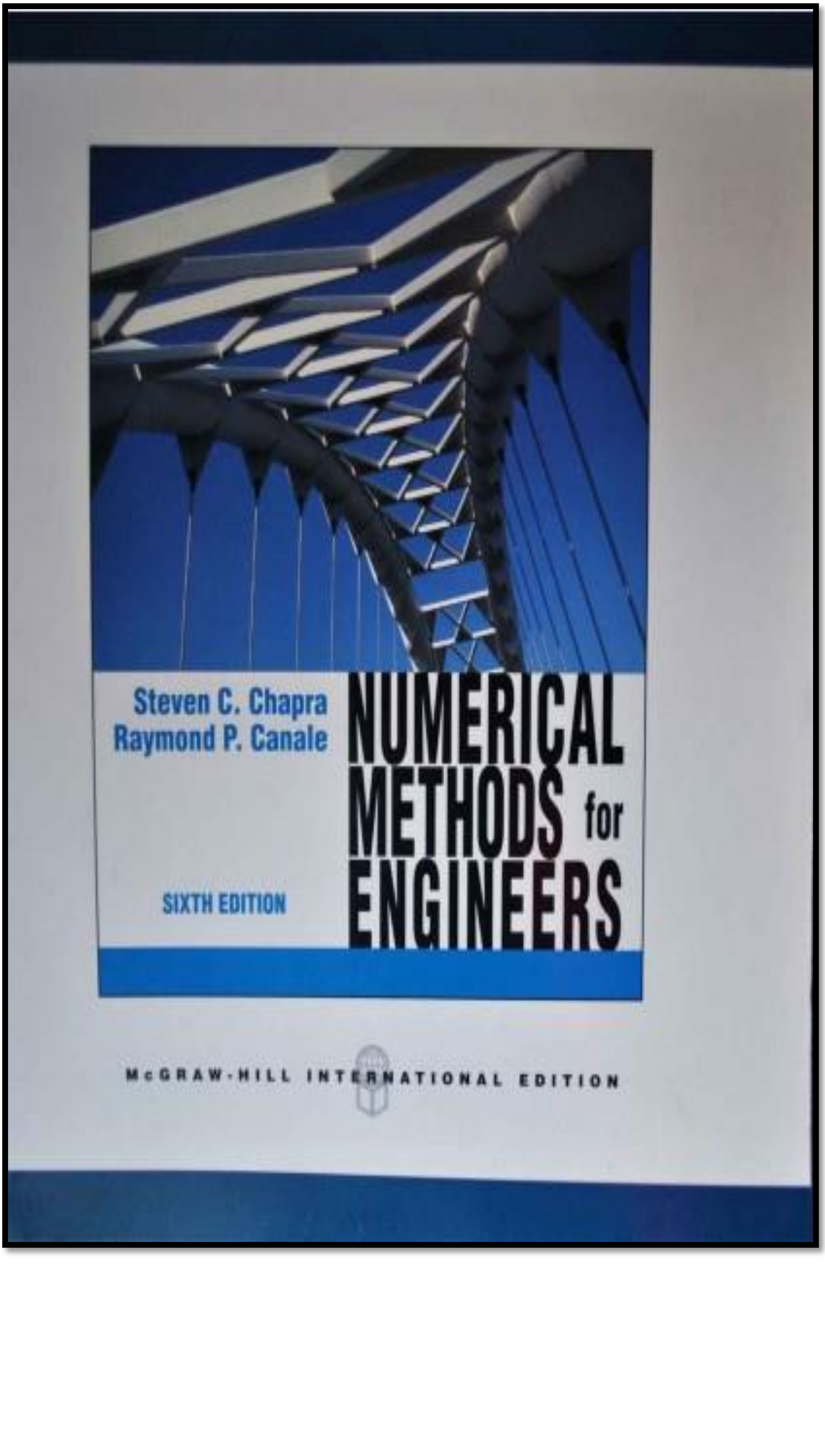

Steven C. Chapra
Raymond P. Canale
NUMERICAL METHODS for ENGINEERS
SIXTH EDITION
McGRAW-HILL INTERNATIONAL EDITION

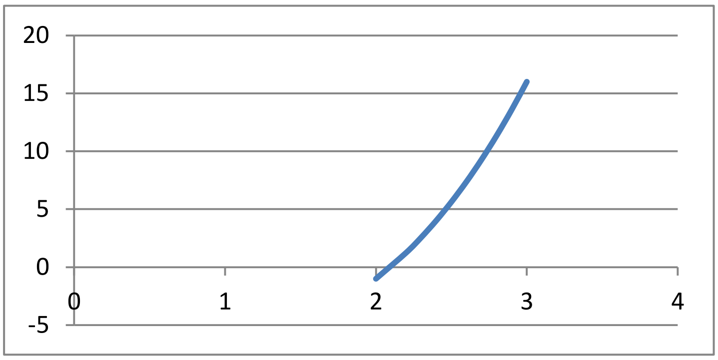
Chapter 5 : Bracketing Methods

- ## 5.1 : Graphical Method

❖ ***Find the root of equation*** $f(x) = x^3 - 2x - 5 = 0$

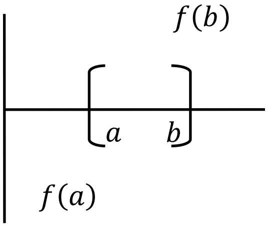
$\Rightarrow f(0) = 0 - 5 = -5$ $\Rightarrow f(1) = 1 - 2 - 5 = -6$

$\Rightarrow f(2) = 8 - 4 - 5 = -1$ $\Rightarrow f(3) = 27 - 6 - 5 = 16$ $\therefore$ *the root lies between* (2 , 3)



f(2.1)	0.061
f(2.2)	1.248
f(2.3)	2.567
f(2.4)	4.024
f(2.5)	5.628
f(2.6)	7.376
f(2.7)	9.283
f(2.8)	11.352
f(2.9)	13.589

- ## <u>5.2 : Bisection Method</u>



$a = the\ negitive\ number$

$b = the\ positive\ number$

$\Rightarrow X_r = \dfrac{a+b}{2}$

$\Rightarrow \mathcal{E}_a = \left|\dfrac{X_r^{new} - X_r^{old}}{X_r^{new}}\right| \times 100$

▶ **3 ways to stop iteration :**

1- Result of function a or b ≅ zero
2- Two X's should equal to each other
3- Error = zero or ≤ 10 % (depend on question)

❖ ***Find the root of equation*** $\boldsymbol{f(x) = x^3 - 2x - 5 = 0}$

$\Rightarrow f(0) = 0 - 5 = negtive$ $\Rightarrow f(1) = 1 - 2 - 5 = negtive$

$\Rightarrow f(2) = 8 - 4 - 5 = negtive$ $\Rightarrow f(3) = 27 - 6 - 5 = positive$

$\therefore a = 2 \quad and \quad b = 3$

$\Rightarrow \boldsymbol{1^{st}\ iteration:}$ $X_1 = \frac{2+3}{2} = 2.5$ $\Rightarrow f(2.5) = (2.5)^3 - 2(2.5) - 5 = 5.625$

$\Rightarrow \boldsymbol{2^{nd}\ iteration:}$ $a = 2\ and\ b = 2.5$ $\Rightarrow X_2 = \frac{2+2.5}{2} = 2.25$ $\Rightarrow f(2.25) = 1.891$

$\Rightarrow \mathcal{E}_a = \left|\frac{2.25 - 2.5}{2.25}\right| \times 100 = 11.111\%$

$\Rightarrow \boldsymbol{3^{rd}\ iteration:}$ $a = 2\ and\ b = 2.25$ $\Rightarrow X_3 = \dfrac{2+2.25}{2} = 2.125$

$\Rightarrow f(2.125) = 0.346$ $\Rightarrow \mathcal{E}_a = \left|\frac{2.125 - 2.25}{2.125}\right| \times 100 = 5.882\%$

$\Rightarrow \boldsymbol{4^{th}\ iteration:}$ $a = 2\ and\ b = 2.125$ $\Rightarrow X_4 = \dfrac{2+2.125}{2} = 2.0625$

$\Rightarrow f(2.063) = -0.346$ $\Rightarrow \mathcal{E}_a = \left|\frac{2.063 - 2.125}{2.063}\right| \times 100 = 3.005\%$

$\Rightarrow \boldsymbol{5^{th}\ iteration:}$ $a = 2.063\ and\ b = 2.125$ $\Rightarrow X_5 = \dfrac{2.063+2.125}{2} = 2.094$

$\Rightarrow f(2.094) = -0.006$ $\Rightarrow \mathcal{E}_a = \left|\frac{2.094 - 2.063}{2.094}\right| \times 100 = 1.480\ \%$

$\Rightarrow \boldsymbol{6^{th}\ iteration:}$ $a = 2.094\ and\ b = 2.125$ $\Rightarrow X_6 = \dfrac{2.094+2.125}{2} = 2.110$

$\Rightarrow f(2.110) = 0.174$ $\Rightarrow \mathcal{E}_a = \left|\frac{2.110 - 2.094}{2.110}\right| \times 100 = 0.758\ \%$

$\Rightarrow \boldsymbol{7^{th}\ iteration:}$ $a = 2.094\ and\ b = 2.110$ $\Rightarrow X_7 = \dfrac{2.094+2.110}{2} = 2.102$

$\Rightarrow f(2.102) = 0.083 \quad \Rightarrow \ \mathcal{E}_a = \left|\frac{2.102-2.110}{2.102}\right| \times 100 = 0.381\ \%$

$\Rightarrow \boldsymbol{8^{th}\ iteration}:\ a = 2.094\ \ and\ \ b = 2.102 \quad \Rightarrow X_8 = \dfrac{2.094 + 2.102}{2} = 2.098$

$\Rightarrow f(2.098) = 0.039 \quad \Rightarrow \ \mathcal{E}_a = \left|\frac{2.098-2.102}{2.098}\right| \times 100 = 0.191\ \%$

$\Rightarrow \boldsymbol{9^{th}\ iteration}:\ a = 2.094\ \ and\ \ b = 2.098 \quad \Rightarrow X_6 = \dfrac{2.094 + 2.098}{2} = 2.096$

$\Rightarrow f(2.096) = 0.174 \quad \Rightarrow \ \mathcal{E}_a = \left|\frac{2.096-2.098}{2.096}\right| \times 100 = 0.095\ \%$

<u>5.2 Page 139</u> **:** $\boldsymbol{f(x) = 4x^3 - 6x^2 + 7x - 2.3}$

$\Rightarrow f(0) = 0 - 2.3 = negtive \qquad \Rightarrow f(1) = 4 - 6 + 7 - 2.3 = positive$

$\therefore root\ lies\ between\ [\,0\,,1\,]$

Iteration	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{X_r^{new}}$	$\boldsymbol{f(X_r)}$	$\boldsymbol{X_r^{old}}$	$\boldsymbol{\mathcal{E}_a}$
1	0	1	0.5	0.2	------	------
2	0	0.5	0.25	-0.868	0.5	100 %
3	0.25	0.5	0.375	-0.308	0.25	33.333 %
4	0.375	0.5	0.438	-0.049	0.375	14.385 %
5	0.438	0.5	0.469	0.076	0.438	6.610 %

- **5.3 : False-Position Method or Rugula Falsi or Linear Interpolation Method :**

$\Rightarrow X_r = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = x\tan x + 1\ in\ [\,2.5\,,3\,]}$

$\Rightarrow f(2.5) = 2.5\tan(2.5) + 1 = -0.868$ $\qquad\Rightarrow f(3) = 3\tan(3) + 1 = 0.572$

$\therefore a = 2.5\ , f(2.5) = -0.868\quad and\ \ b = 3\ , f(3) = 0.572$

$\Rightarrow \boldsymbol{1^{st}\ iteration}:\quad X_1 = \frac{2.5\,(\,0.572\,) - 3\,(\,-0.868\,)}{0.572 + 0.868} = 2.801$

$\Rightarrow \boldsymbol{2^{nd}\ iteration}:\quad a = 2.5\ , f(2.5) = -0.868\quad and\ \ b = 2.801\ , f(2.801) = 0.007$

$X_2 = \frac{2.5\,(\,0.007\,) - 2.801\,(\,-0.868\,)}{0.007+0.868} = 2.799 \qquad \Rightarrow \mathcal{E}_a = \left|\frac{2.799-2.801}{2.799}\right| \times 100 = 0.071\%$

$\Rightarrow \boldsymbol{3^{rd}\ iteration}:\quad a = 2.5\ , f(2.5) = -0.868\quad and\ \ b = 2.799\ , f(2.799) = 0.002$

$X_3 = \frac{2.5\,(\,0.002\,) - 2.799\,(\,-0.868\,)}{0.002+0.868} = 2.798 \qquad \Rightarrow \mathcal{E}_a = \left|\frac{2.798-2.799}{2.798}\right| \times 100 = 0.036\%$

$\Rightarrow \boldsymbol{4^{th}\ iteration}:\quad a = 2.798\ , f(2.798) = -0.001\quad and\ \ b = 2.799\ , f(2.799) = 0.002$

$X_4 = \frac{2.798\,(\,0.002\,) - 2.799\,(\,-0.001\,)}{0.002+0.001} = 2.798 \qquad \because \boldsymbol{X_4 = X_3\ \ we\ stop\ iteration}$

5.11 Page 139 : $\boldsymbol{f(x) = x^{3.5} - 80}$

$\Rightarrow f(2) = 2^{3.5} - 80 = -68.686$ $\qquad\Rightarrow f(5) = 5^{3.5} - 80 = 199.508$

Iteration	a	$f(a)$	b	$f(b)$	X_r^{new}	$f(X_r)$	X_r^{old}	$\mathcal{E}_a$
1	2	-68.686	5	199.508	2.768	-44.716	------	------
2	2.768	-44.716	5	199.508	3.177	-22.844	2.768	12.874 %
3	3.177	-22.844	5	199.508	3.364	-10.177	3.177	5.559 %
4	3.364	-10.177	5	199.508	3.443	-4.268	3.364	2.295 %
5	3.443	-4.268	5	199.508	3.476	-1.728	3.443	0.949 %
6	3.476	-1.728	5	199.508	3.489	-0.660	3.476	0.373 %
7	3.489	-0.660	5	199.508	3.494	-0.270	3.489	0.143 %
8	3.494	-0.270	5	199.508	3.496	-0.106	3.494	0.057 %
9	3.496	-0.106	5	199.508	3.497	-0.045	3.496	0.029 %
10	3.497	-0.045	5	199.508	3.497	$\because \boldsymbol{X_{10} = X_9\ \ we\ stop\ iteration}$		

Chapter 6 : Open Methods

- ## 6.1 : Simple Fixed Method Iteration Method or Method Successive Approximation

$\Rightarrow X_1 = \phi\, X_0 \quad \Rightarrow |\phi'\,(X)\,| < 1\,,\ [\,a\,,b\,]$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = x^3 + x^2 - 1}$

$\Rightarrow f(0) = 0 + 0 - 1 = 0 \qquad \Rightarrow f(1) = 1 + 1 - 1 = 1 \quad \therefore root\ lies\ between\ [\,0\,,1\,]$

$$\boldsymbol{f(x) = x^3 + x^2 - 1}$$

❶ $\Rightarrow \boldsymbol{x^3 = x^2 - 1}$

$\Rightarrow \boldsymbol{x = (1 - x^2)^{\frac{1}{3}}}$

❷ $\Rightarrow \boldsymbol{x^2 = x^3 - 1}$

$\Rightarrow \boldsymbol{x = (1 - x^3)^{\frac{1}{2}}}$

❸ $\Rightarrow \boldsymbol{x^2(x+1) - 1 = 0}$

$\Rightarrow \boldsymbol{x^2(x+1) = 1}$

$\Rightarrow \boldsymbol{x^2 = \frac{1}{x+1}}$

$\Rightarrow \boldsymbol{x = \frac{1}{\sqrt{x+1}}}$

❶ : $\phi'\,(X) = \frac{1}{3}(1 - x^2)^{\frac{1}{3}-1}\,(\,-2x\,) = -\frac{2}{3}\frac{x}{(1 - x^2)^{\frac{2}{3}}} \qquad \Rightarrow \boldsymbol{we\ assume\ x} = 0.9$

$= \left|-\frac{2}{3}\frac{0.9}{(1 - 0.9^2)^{\frac{2}{3}}}\right| = 1.815\ > 1 \quad \boldsymbol{so\ we\ can't\ use\ this\ equation}$

❷ : $\boldsymbol{we\ can't\ use\ this\ equation\ because\ same\ procedure\ of\ last\ one}$

❸ : $\phi'\,(X) = \frac{-1}{2}(1 + x)^{\frac{-3}{2}} = \frac{-1}{2(x+1)^{\frac{3}{2}}} \qquad \Rightarrow \boldsymbol{we\ assume\ x} = 0.9$

$= \left|\frac{-1}{2(0.9 + 1)^{\frac{3}{2}}}\right| = 0.191\ < 1 \quad \boldsymbol{so\ we\ can\ use\ this\ equation}$

$\Rightarrow \boldsymbol{1^{st}\ iteration}: \quad \Rightarrow X_0 = 0.5 \quad \Rightarrow X_1 = \frac{1}{\sqrt{1 + 0.5}} = 0.816$

$\Rightarrow \boldsymbol{2^{nd}\ iteration}: \quad \Rightarrow X_2 = \frac{1}{\sqrt{1 + 0.816}} = 0.742$

$\Rightarrow \boldsymbol{3^{rd}\ iteration}: \quad \Longrightarrow X_3 = \dfrac{1}{\sqrt{1+0.742}} = 0.758$

$\Rightarrow \boldsymbol{4^{th}\ iteration}: \quad \Longrightarrow X_4 = \dfrac{1}{\sqrt{1+0.758}} = 0.754$

$\Rightarrow \boldsymbol{5^{th}\ iteration}: \quad \Longrightarrow X_5 = \dfrac{1}{\sqrt{1+0.754}} = 0.755$

$\Rightarrow \boldsymbol{6^{th}\ iteration}: \quad \Longrightarrow X_5 = \dfrac{1}{\sqrt{1+0.755}} = 0.755 \qquad \because \boldsymbol{X_5 = X_6\ \ we\ stop\ iteration}$

❖ ***Find the root of equation*** $\boldsymbol{f(x) = 2x - \log_{10} x - 7 = 0}$ ***and error*** $< 7\,\%$

$\Rightarrow f(0) = error \qquad \Rightarrow f(1) = 2 - \log_{10} 1 - 7 = -5$

$\Rightarrow f(2) = 4 - \log_{10} 2 - 7 = -3.301 \qquad \Rightarrow f(3) = 6 - \log_{10} 3 - 7 = -1.477$

$\Rightarrow f(4) = 8 - \log_{10} 4 - 7 = 0.398 \qquad \therefore root\ lies\ between\ [\,3\,,4\,]$

$$\boldsymbol{f(x) = 2x - \log_{10} x - 7 \Rightarrow x = \frac{1}{2}[\log_{10} x + 7]}$$

❶

$\Longrightarrow \log_{10} x = \log_e x \ .\log_{10} e$

$\Longrightarrow \log_e x = \ln x$

$\Longrightarrow \dfrac{d}{dx}(\log_{10} x)$

$= \log_{10} e \ .\dfrac{1}{x} \quad = \dfrac{0.434}{x}$

❷

$\Longrightarrow -\log_{10} x = 7 - 2x$

$\Longrightarrow \log_{10} x = 2x - 7$

$\Longrightarrow x = \text{Antilog}(2x - 7)$

❶ : $\quad \phi(X) = \dfrac{1}{2}[7 + \log_{10} x] \qquad \Longrightarrow \phi'(X) = \dfrac{1}{2}\left[\dfrac{0.434}{x}\right] \qquad \Rightarrow$ ***we assume*** $\boldsymbol{x} = 3.9$

$= \left|\dfrac{1}{2} \times \dfrac{0.434}{3.9}\right| = 0.056 < 1$ ***so we can use this equation***

❷ : ***we can't use this equation***

$\Rightarrow \boldsymbol{1^{st}\ iteration}: \quad \Longrightarrow X_0 = 3.5 \quad \Longrightarrow X_1 = \dfrac{1}{2}[7 + \log_{10} 3.5] = 3.772$

$\Rightarrow \boldsymbol{2^{nd}\ iteration}: \quad \Longrightarrow X_2 = \dfrac{1}{2}[7 + \log_{10} 3.772] = 3.788$

$\Rightarrow \mathcal{E}_a = \left|\dfrac{3.788 - 3.772}{3.788}\right| \times 100 = 0.430\%$

$\Rightarrow$ $\boldsymbol{3^{rd}}$ ***iteration*** : $\Longrightarrow X_3 = \frac{1}{2}[7 + \log_{10} 3.788] = 3.789$

$\Rightarrow \mathcal{E}_a = \left|\frac{3.789 - 3.788}{3.789}\right| \times 100 = 0.026\%$

$\Rightarrow$ $\boldsymbol{4^{th}}$***iteration*** : $\Longrightarrow X_4 = \frac{1}{2}[7 + \log_{10} 3.789] = 3.789$

$\because$ $\boldsymbol{X_4 = X_3}$ ***we stop iteration***

❖ ***Find the root of equation*** $\boldsymbol{f(x) = \cos x - x^2 - x = 0}$

$\Rightarrow f(0) = \cos 0 - 0 = 1$ $\Rightarrow f(1) = \cos 1 - 1 - 1 = -1$ $\therefore$ *root lies between* $[\,0\,, 1\,]$

$$\boldsymbol{f(x) = \cos x - x^2 - x}$$

➊
$\Longrightarrow \cos x = x^2 + x$
$\Longrightarrow \cos x = x\,(x + 1)$
$\Longrightarrow x = \frac{\cos x}{(x + 1)}$

➋
$\Longrightarrow \cos x = x^2 + x$
$\Longrightarrow \cos x = x\,(x + 1)$
$\Longrightarrow x = \cos^{-1}[x\,(x + 1)]$

➊ : $\phi'(X) = \frac{(x + 1)(-\sin x) - \cos x\ (1)}{(x + 1)^2}$ $\Rightarrow$ ***we assume*** $\boldsymbol{x} = 0.9$

$= \left|\frac{(0.9 + 1)(-\sin 0.9) - \cos 0.9\ (1)}{(0.9 + 1)^2}\right| = 0.285 < 1$ ***so we can use this equation***

➋ : ***we can't use this equation***

Iteration	X_n	X_r	$\mathcal{E}_a$
1	0.5	0.585	------
2	0.585	0.526	11.217 %
3	0.526	0.567	7.231 %
4	0.567	0.538	5.390 %
5	0.538	0.558	3.584 %
6	0.558	0.544	2.574 %
7	0.544	0.554	1.805 %
8	0.554	0.547	1.280 %
9	0.547	0.552	0.906 %
10	0.552	0.549	0.546 %
11	0.549	0.551	0.363 %
12	0.551	0.549	0.364 %

- ## 6.2 : Newton's Raphson Method

$$\Rightarrow X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

$let\ \phi(X) = X - \frac{f(X)}{f'(X)}$ $\Rightarrow for\ any\ iteration\ methods\ |\phi'(X)| < 1$

$$\Rightarrow \frac{d}{dx}\left(\phi(X)\right) = \frac{d}{dx}\left(X - \frac{f(X)}{f'(X)}\right)$$

$$\Rightarrow \phi'(X) = 1 - \left[\frac{f'(X)\,f'(X) - f(X)\,f''(X)}{\left(f'(X)\right)^2}\right] = \frac{f(X)\,f''(X)}{\left(f'(X)\right)^2}$$

$$\Rightarrow \left|\frac{f(X)\,f''(X)}{\left(f'(X)\right)^2}\right| < 1 \quad \Rightarrow |f(X)\,f''(X)| < \left(f'(X)\right)^2$$

$$\Rightarrow \boldsymbol{Error}: \ E_{t,i+1} = \left|-\frac{f''(X_r)}{2f'(X_r)} \times E_{t,i}{}^2\right| \quad \Rightarrow \boldsymbol{i = 0}$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = x^3 + x - 1}$

$\Rightarrow f(0) = 0 + 0 - 1 = -1$ $\Rightarrow f(1) = 1 + 1 - 1 = 1$ $\therefore root\ lies\ between\ [0, 1]$

n	X_n	$f(x)$	$f'(x)$	$f''(x)$
------	------	$x^3 + x - 1$	$3x^2 + 1$	$6x$
0	0.5	-0.375	1.750	3
1	0.714	0.078	2.529	4.284
2	0.683	0.002	2.399	4.098
3	0.682	-0.001	2.395	4.092

$\Rightarrow \boldsymbol{1^{st}\ iteration}: \Rightarrow X_1 = X_0 - \frac{f(X_0)}{f'(X_0)} = 0.5 - \frac{-0.375}{1.75} = 0.714$

$$\Rightarrow E_{t,0+1} = \left|-\frac{f''(X_1)}{2f'(X_1)} \times E_{t,0}{}^2\right| = \left|-\frac{4.284}{2(2.529)} \times 0.5^2\right| = 0.212$$

$\Rightarrow \boldsymbol{2^{nd}\ iteration}: \Rightarrow X_2 = X_1 - \frac{f(X_1)}{f'(X_1)} = 0.714 - \frac{0.078}{2.529} = 0.683$

$$\Rightarrow E_{t,2} = \left|-\frac{f''(X_2)}{2f'(X_2)} \times E_{t,1}{}^2\right| = \left|-\frac{4.098}{2(2.399)} \times 0.212^2\right| = 0.038$$

$\Rightarrow \boldsymbol{3^{rd}\ iteration}: \Rightarrow X_3 = X_2 - \frac{f(X_2)}{f'(X_2)} = 0.683 - \frac{0.002}{2.399} = 0.682$

$$\Rightarrow E_{t,3} = \left|-\frac{f''(X_3)}{2f'(X_3)} \times E_{t,2}{}^2\right| = \left|-\frac{4.092}{2(2.395)} \times 0.038^2\right| = 0.001$$

$\Rightarrow \boldsymbol{4^{th}\ iteration}: \Rightarrow X_4 = X_3 - \frac{f(X_3)}{f'(X_3)} = 0.682 - \frac{-0.001}{2.395} = 0.682$

$\because\ X_4 = X_3$ ***we stop iteration***

- <u>**6.3 : The Secant Method**</u>

$$\Rightarrow X_{n+1} = X_n - \frac{f(X_n)(X_{n-1} - X_n)}{f(X_{n-1}) - f(X_n)}$$

$$\Rightarrow \boldsymbol{n = 0} \Rightarrow X_1 = X_0 - \frac{f(X_0)(X_{-1} - X_0)}{f(X_{-1}) - f(X_0)}$$

$$\Rightarrow \boldsymbol{n = 1} \Rightarrow X_2 = X_1 - \frac{f(X_1)(X_0 - X_1)}{f(X_0) - f(X_1)}$$

❖ ***Find the root of equation*** $\boldsymbol{f(x) = -x^2 + x + 0.75}$

$\Rightarrow f(0) = -0 + 0 + 0.75 = 0.75 \qquad \Rightarrow f(1) = -1 + 1 + 0.75 = 0.75$

$\Rightarrow f(2) = -4 + 2 + 0.75 = -1.25 \qquad \therefore$ *root lies between* $[1, 2]$

By Newton's Raphson Method : $X_0 = X_{-1} - \dfrac{f(X_{-1})}{f'(X_{-1})} = 1.9 - \dfrac{-0.960}{-2.8} = 1.557$

$\boldsymbol{n}$	$\boldsymbol{X_n}$	$\boldsymbol{f(x)}$	$\boldsymbol{f'(x)}$
------	------	$\boldsymbol{-x^2 + x + 0.75}$	$-2x + 1$
-1	1.9	-0.960	-2.8
0	1.557	-0.117	-2.114

By Secant Method :

1	1.509	-0.018	-2.018
2	1.5	0	-2
3	1.5	$\because\ X_3 = X_2$ ***we stop iteration***	

$\Rightarrow$ $\boldsymbol{1^{st}}$ ***iteration*** : $\Rightarrow X_1 = X_0 - \dfrac{f(X_0)(X_{-1} - X_0)}{f(X_{-1}) - f(X_0)}$

$$= 1.557 - \frac{(-0.117)(1.9 - 1.557)}{-0.960 + 0.117} = 1.509$$

$\Rightarrow$ $\boldsymbol{2^{nd}}$ ***iteration*** : $\Rightarrow X_2 = X_1 - \dfrac{f(X_1)(X_0 - X_1)}{f(X_0) - f(X_1)}$

$$= 1.509 - \frac{(-0.018)(1.557 - 1.509)}{-0.117 + 0.018} = 1.5$$

$\Rightarrow$ $\boldsymbol{3^{rd}}$ ***iteration*** : $\Rightarrow X_3 = X_2 - \dfrac{f(X_2)(X_1 - X_2)}{f(X_1) - f(X_2)}$

$$= 1.5 - \frac{(0)(1.509 - 1.5)}{-0.018 - 0} = 1.5$$

6.2 Page 171 : $f(x) = 2x^3 - 11.7x^2 + 17.7x - 5$

C) by Newton's Raphson Method : $x_0 = 3$

n	X_n	$f(x)$	$f'(x)$	$f''(x)$	$E_{t,i+1}$
-----	------	$2x^3 - 11.7x^2 + 17.7x - 5$	$6x^2 - 23.4x + 17.7$	$12x - 23.4$	------
0	3	-3.2	1.5	3.6	------
1	5.133	48.072	55.674	38.196	1.029
2	4.270	12.963	27.179	27.840	0.527
3	3.793	2.949	15.264	22.116	0.382
4	3.6	0.4	11.22	19.8	0.337
5	3.564	0.009	10.515	19.368	0.347
6	3.563	-0.002	10.496	19.346	0.320
7	3.563	$\because X_7 = X_6$ *we stop iteration*			

D) by Secant Method : $x_{-1} = 3$, $x_0 = 4$ *and error* $< 0.5\%$

n	X_n	$f(x)$	$\mathcal{E}_a$
------	------	$2x^3 - 11.7x^2 + 17.7x - 5$	------
-1	3	-3.2	------
0	4	6.6	25 %
1	3.327	-1.966	20.228 %
2	3.481	-0.798	4.424 %
3	3.586	0.245	2.928 %
4	3.561	-0.023	0.702 %
5	3.563	-0.002	0.056 %

6.9 Page 172 : $f(x) = 0.95x^3 - 5.9x^2 + 10.9x - 6$

B) by Newton's Raphson Method : $x_0 = 3.5$

n	X_n	$f(x)$	$f'(x)$	$f''(x)$	$E_{t,i+1}$
-----	------	$0.95x^3 - 5.9x^2 + 10.9x - 6$	$2.85x^2 - 11.8x + 10.9$	$5.7x - 11.8$	------
0	3.5	0.606	4.513	8.150	------
1	3.366	0.072	3.472	7.386	13.030
2	3.345	0.001	3.318	7.267	185.925
3	3.345	$\because X_3 = X_2$ *we stop iteration*			

C) by Secant Method : $x_{-1} = 2.5$, $x_0 = 3.5$

n	X_n	$f(x)$	$\mathcal{E}_a$
------	------	$0.95x^3 - 5.9x^2 + 10.9x - 6$	------
-1	2.5	-0.781	------
0	3.5	0.606	28.571 %
1	3.063	-0.667	14.267 %
2	3.292	-0.165	6.956 %
3	3.367	0.076	2.228 %
4	3.343	-0.005	0.718 %
5	3.344	-0.002	0.030 %
6	3.345	0.001	0.030 %
7	3.345	$\because X_7 = X_6$ *we stop iteration*	

- <u>6.6 : The System of non linear equations</u>

$\Rightarrow U(x,y) = x^2 + xy - 10 = 0 \qquad \Rightarrow V(x,y) = y + 3xy^2 - 57 = 0$

$\Rightarrow$ ***by fixed point iteration*** :

$\Rightarrow x = U(x,y) = x^2 + xy - 10 = 0 \qquad \Rightarrow xy = 10 - x^2$

$\Rightarrow x = \frac{10-x^2}{y} \qquad \Rightarrow x_{n+1} = \frac{10-{x_n}^2}{y_n}$

$\Rightarrow y = V(x,y) = y + 3xy^2 - 57 = 0 \qquad \Rightarrow y = 57 - 3xy^2$

$\Rightarrow x = \frac{10-x^2}{y} \qquad \Rightarrow y_{n+1} = 57 - 3x_n{y_n}^2$

$\Rightarrow x_0 = 1.5 \qquad \Rightarrow y_0 = 3.5$

$\Rightarrow \mathbf{1^{st}}$ ***iteration*** : $\Rightarrow x_{0+1} = \frac{10-{x_0}^2}{y_0} = \frac{10-1.5^2}{3.5} = 2.214$

$\Rightarrow y_{0+1} = 57 - 3x_1{y_0}^2 = 57 - 3(2.214)(3.5)^2 = -24.365$

$\Rightarrow \mathbf{2^{nd}}$ ***iteration*** : $\Rightarrow x_2 = \frac{10-{x_1}^2}{y_1} = \frac{10-2.214^2}{-24.365} = -0.209$

$\Rightarrow y_2 = 57 - 3x_2{y_1}^2 = 57 - 3(-0.209)(-24.365)^2 = 429.221$

$\Rightarrow \mathbf{3^{rd}}$ ***iteration*** : $\Rightarrow x_3 = \frac{10-{x_2}^2}{y_2} = \frac{10-(-0.209)^2}{429.221} = 0.023$

$\Rightarrow y_3 = 57 - 3x_3{y_2}^2 = 57 - 3(0.023)(429.221)^2 = -12654.916$

$\Rightarrow$ ***by Newton Raphson*** :

$$\Rightarrow x_{i+1} = x_i - \frac{u_i\frac{\partial V_i}{\partial y} - v_i\frac{\partial U_i}{\partial y}}{\frac{\partial U_i}{\partial x}\frac{\partial V_i}{\partial y} - \frac{\partial U_i}{\partial y}\frac{\partial V_i}{\partial x}} \qquad \Rightarrow y_{i+1} = y_i - \frac{v_i\frac{\partial U_i}{\partial x} - u_i\frac{\partial V_i}{\partial x}}{\frac{\partial U_i}{\partial x}\frac{\partial V_i}{\partial y} - \frac{\partial U_i}{\partial y}\frac{\partial V_i}{\partial x}}$$

$\Rightarrow U(x,y) = x^2 + xy - 10 = 0 \qquad \Rightarrow V(x,y) = y + 3xy^2 - 57 = 0$

$\Rightarrow x_0 = 1.5 \qquad \Rightarrow y_0 = 3.5$

$\Rightarrow \left|\frac{\partial U_0}{\partial x}\right| = 2x + y = 2(1.5) + 3.5 = 6.5 \qquad \Rightarrow \left|\frac{\partial U_0}{\partial y}\right| = x = 1.5$

$\Rightarrow \left|\frac{\partial V_0}{\partial x}\right| = 3xy^2 = 3(3.5)^2 = 36.75 \qquad \Rightarrow \left|\frac{\partial V_0}{\partial y}\right| = 1 + 6xy = 1 + 6(1.5)(3.5) = 32.5$

$\Rightarrow u_0 = (1.5)^2 + 1.5(3.5) - 10 = -2.5 \qquad \Rightarrow v_0 = 3.5 + 3(1.5)(3.5)^2 - 57 = 1.625$

$\Rightarrow \frac{\partial U_i}{\partial x}\frac{\partial V_i}{\partial y} - \frac{\partial U_i}{\partial y}\frac{\partial V_i}{\partial y} = 6.5\,(32.5) - 1.5\,(36.75) = 156.125$

$$\Rightarrow x_{i+1} = 1.5 - \frac{(-2.5)(32.5) - (1.625)(1.5)}{156.125} = 2.036$$

$$\Rightarrow y_{i+1} = 3.5 - \frac{(1.625)(6.5) - (-2.5)(36.75)}{156.125} = 2.844$$

Chapter 21 : Newton-Cotes Integration Formulas

- **21.1 : Trapezoidal Rule**

$$\Rightarrow \int_{X_0}^{X_0+nh} f(x)dx = \frac{h}{2}[(Y_0 + Y_n) + 2(Y_1 + Y_2 + \dots\dots + Y_{n-1})]$$

$\Rightarrow n = number\ of\ interval$ $\Rightarrow h = size\ of\ interval$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_4^{5.2} \frac{1}{x} dx}$

$= [\ln x]_4^{5.2} = \ln 5.2 - \ln 4 = 0.262$ " *this solve called exaxt or normal rule*"

$\Rightarrow X_0 + nh = 5.2 \quad \Rightarrow X_0 = 4 \quad \Rightarrow 4 + nh = 5.2 \quad \Rightarrow nh = 1.2$

$\Rightarrow let\ h = 0.2 \quad \therefore n = \frac{1.2}{0.2} = 6$

X	4	4.2	4.4	4.6	4.8	5	5.2
$f(x)$	0.25	0.238	0.227	0.217	0.208	0.2	0.192
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_{n=6}$

$$\Rightarrow \int_4^{5.2} \frac{1}{x} dx = \frac{0.2}{2}[(0.25 + 0.192) + 2(0.238 + 0.227 + 0.217 + 0.208 + 0.2)] = 0.262$$

$\Rightarrow Error = Exact - Calculated = 0.262 - 0.262 = 0$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_{0.2}^{1.4} (sinx - \ln x + e^x)\ dx}$

$= [-cosx - (xlnx - x) + e^x]_2^{1.4}$

$= [-cos1.4 - (1.4ln1.4 - 1.4) + e^{1.4}] - [-cos0.2 - (0.2ln0.2 - 0.2) + e^{0.2}] = 4.051$

$\Rightarrow X_0 + nh = 1.4 \quad \Rightarrow X_0 = 0.2 \quad \Rightarrow 0.2 + nh = 1.4 \quad \Rightarrow nh = 1.2$

$\Rightarrow let\ h = 0.2 \quad \therefore n = \frac{1.2}{0.2} = 6$

X	0.2	0.4	0.6	0.8	1	1.2	1.4
$f(x)$	3.030	2.798	2.898	3.166	3.560	4.070	4.704
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_{n=6}$

$$= \frac{0.2}{2}[(3.030 + 4.704) + 2(2.798 + 2.898 + 3.166 + 3.560 + 4.070)] = 4.072$$

$\Rightarrow Error = Exact - Calculated = 4.051 - 4.072 = 0.021$

- **21.2 : Simpson's Rule**

1) $\frac{1}{3}$ ***Rule*** : $\int_{X_0}^{X_0+nh} f(x)dx = \frac{h}{3}[(Y_0 + Y_n) + 4(Y_1 + Y_3 + Y_{odd}) + 2(Y_2 + Y_4 + Y_{even})]$

$\Rightarrow Intervales\ should\ be\ even\ number\ \ \boldsymbol{n = even}$

2) $\frac{3}{8}$ ***Rule*** : $\int_{X_0}^{X_0+nh} f(x)dx = \frac{3h}{8}[(Y_0 + Y_n) + 3(Y_1 + Y_2 + Y_4 + Y_5 + Y_7 + \cdots) + 2(Y_3 + Y_6 + Y_9 + Y_{12} + \cdots)]$

$\Rightarrow Intervales\ should\ be\ multiple\ of\ 3\ \ \boldsymbol{n = \times 3}$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_4^{5.2} \frac{1}{x} dx}$

$= [\ln x]_4^{5.2} = \ln 5.2 - \ln 4 = 0.262 \Rightarrow let\ h = 0.2 \quad \Rightarrow n = 6$

X	4	4.2	4.4	4.6	4.8	5	5.2
$f(x)$	0.25	0.238	0.227	0.217	0.208	0.2	0.192
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_{n=6}$

$$\Rightarrow \int_4^{5.2} \frac{1}{x} dx = \frac{0.2}{3}[(0.25 + 0.192) + 4(0.238 + 0.217 + 0.2) + 2(0.227 + 0.208)] = 0.262$$

$$\Rightarrow \int_4^{5.2} \frac{1}{x} dx = \frac{3 \times 0.2}{8}[(0.25 + 0.192) + 2(0.217) + 3(0.238 + 0.227 + 0.208 + 0.2)] = 0.262$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_{0.2}^{1.4} (sinx - \ln x + e^x)\ dx}$

$= [-cosx - (xlnx - x) + e^x]_2^{1.4} = 4.051$

$\Rightarrow X_0 + nh = 1.4 \quad \Rightarrow X_0 = 0.2 \quad \Rightarrow 0.2 + nh = 1.4 \quad \Rightarrow nh = 1.2$

$\Rightarrow let\ h = 0.2 \quad \therefore n = \frac{1.2}{0.2} = 6$

X	0.2	0.4	0.6	0.8	1	1.2	1.4
$f(x)$	3.030	2.798	2.898	3.166	3.560	4.070	4.704
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_{n=6}$

$$= \frac{0.2}{3}[(3.030 + 4.704) + 4(2.798 + 3.166 + 4.070) + 2(2.898 + 3.560)] = 4.052$$

$\Rightarrow Error = Exact - Calculated = 4.051 - 4.052 = 0.001$

$$= \frac{3 \times 0.2}{8}[(3.030 + 4.704) + 2(3.166) + 3(2.798 + 2.898 + 3.560 + 4.070)] = 4.053$$

$\Rightarrow Error = Exact - Calculated = 4.051 - 4.053 = 0.002$

- **21.3 : Integration With Un Equal Segments**

$$\Rightarrow I = (b-a)\frac{f(a)+f(b)}{2}$$

21.13 Page 628 : $\boldsymbol{f(x) = 2\, e^{-1.5x}}$

X	0	0.05	0.15	0.25	0.35	0.475	0.6
$f(x)$	2	1.8555	1.5970	1.3746	1.1831	0.9808	0.8131
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_{n=6}$

A) by Analytical mean : $\boldsymbol{a = 0}$ **,** $\boldsymbol{b = 0.6}$

$$\Rightarrow I = (b-a)\frac{f(a)+f(b)}{2} = (0.05-0)\frac{2+1.8555}{2} + (0.25-0.15)\frac{1.5970+1.3746}{2} + (0.475-0.35)\frac{1.1831+0.9808}{2} + (0.6-0.475)\frac{0.9808+0.8131}{2} = 0.492$$

$$\Rightarrow \int_0^{0.6} 2\, e^{-1.5x} dx = 2\left[\frac{e^{-1.5x}}{-1.5}\right]_0^{0.6} = 0.791$$

$$\Rightarrow Error = Exact - Calculated = 0.791 - 0.492 = 0.299$$

- $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_0^4 (1 - e^{-2x}) dx}$

A) Consider equal interval problem using trapezoidal rule with 8 intervals : $\boldsymbol{a = 0}$ **,** $\boldsymbol{b = 0.6}$

$$\Rightarrow X_0 + nh = 4 \quad \Rightarrow X_0 = 0 \quad \Rightarrow nh = 4 \quad \Rightarrow n = 8 \quad \therefore h = \frac{4}{8} = 0.5$$

X	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	0	0.632	0.865	0.950	0.982	0.993	0.998	0.999	1
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8

$$\Rightarrow \int_0^4 (1 - e^{-2x}) dx$$

$$= \frac{0.5}{2}[(0+1) + 2(0.632 + 0.865 + 0.95 + 0.982 + 0.993 + 0.998 + 0.999)] = 3.460$$

B) Consider un equal interval problem : $\boldsymbol{x = 0, 0.1, 0.5, 1.2, 1.4, 2, 2.8, 2.9, 3.2, 3.8, 4}$

X	0	0.1	0.5	1.2	1.4	2	2.8	2.9	3.2	3.8	4
$f(x)$	0	0.181	0.632	0.909	0.939	0.982	0.996	0.997	0.998	0.999	1
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}

$$\Rightarrow I = (b-a)\frac{f(a)+f(b)}{2}$$

$$= (0.1-0)\frac{0+0.181}{2} + (1.2-0.5)\frac{0.632+0.909}{2} + (1.4-2)\frac{0.939+0.982}{2}$$

$$+(2.9-2.8)\frac{0.996+0.997}{2}+(3.8-3.2)\frac{0.998+0.999}{2}+(4-3.8)\frac{1+0.999}{2}=2.023$$

C) Check your answer by exact value :

$$\Rightarrow \int_0^4 (1-\ e^{-2x})dx=\left[x-\frac{e^{-2x}}{-2}\right]_0^4=3.5$$

$$\Rightarrow Error\ for\ A=Exact-Calculated=3.5-3.460=0.04$$

$$\Rightarrow Error\ for\ B=Exact-Calculated=3.5-2.023=1.297$$

Chapter 22 : Integration of Equations

- **22.1 : Newton's Cotes Algorithms of Equations**

$$\Rightarrow \int_a^b f(x)\,dx = (b-a)f\left(\frac{a+b}{2}\right)$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_1^2 x dx}$

$$\Rightarrow \int_1^2 xdx = \left[\frac{x^2}{2}\right]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} = 1.5$$

$$\Rightarrow \int_1^2 xdx = (2-1)f\left(\frac{2+1}{2}\right) = 1 \times f\left(\frac{3}{2}\right) = 1.5$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_0^1 e^{-x^2} dx}$

$$\Rightarrow \int_0^1 e^{-x^2} dx = 0.779$$

$$\Rightarrow \int_0^1 e^{-x^2} dx = (1-0)f\left(\frac{1+0}{2}\right) = 1 \times f\left(\frac{1}{2}\right) = 0.779$$

- ## 22.2 : Romberg Integration

$\Rightarrow \boldsymbol{1^{st}\ iteration:}\quad \Rightarrow I_a = I_2 + \left[\frac{I_2 - I_1}{3}\right] \quad \Rightarrow \boldsymbol{n = 2}$

$\Rightarrow \boldsymbol{2^{nd}\ iteration:}\quad \Rightarrow I_b = I_3 + \left[\frac{I_3 - I_2}{3}\right] \quad \Rightarrow \boldsymbol{n = 4}$

$\Rightarrow \boldsymbol{3^{rd}\ iteration:}\quad \Rightarrow I_c = I_4 + \left[\frac{I_4 - I_3}{3}\right] \quad \Rightarrow \boldsymbol{n = 8}$

$\Rightarrow \boldsymbol{4^{th}\ iteration:}\quad \Rightarrow I_d = I_5 + \left[\frac{I_5 - I_4}{3}\right] \quad \Rightarrow \boldsymbol{n = 16}$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_0^1 \frac{1}{1+x}dx}$

$= [\log(1+x)]_0^1 \quad or \quad [\ln(1+x)]_0^1 \quad = 0.693$

$\Rightarrow X_0 + nh = 1 \quad \Rightarrow X_0 = 0 \quad \Rightarrow nh = 1 \quad \Rightarrow n = 2 \quad \therefore h = \frac{1}{2} = 0.5$

X	**0**	**0.5**	**1**
$\boldsymbol{f(x)}$	1	0.667	0.5
Y	Y_0	Y_1	Y_2

by trapazoidal rule

$$\Rightarrow I_1 = \int_0^1 \frac{1}{1+x}dx = \frac{0.5}{2}[(1+0.5) + 2(0.667)] = 0.709$$

$\Rightarrow X_0 + nh = 1 \quad \Rightarrow X_0 = 0 \quad \Rightarrow nh = 1 \quad \Rightarrow n = 4 \quad \therefore h = \frac{1}{4} = 0.25$

X	**0**	**0.25**	**0.5**	**0.75**	**1**
$\boldsymbol{f(x)}$	1	0.8	0.667	0.571	0.5
Y	Y_0	Y_1	Y_2	Y_3	Y_4

$$\Rightarrow I_2 = \int_0^1 \frac{1}{1+x}dx = \frac{0.25}{2}[(1+0.5) + 2(0.8 + 0.667 + 0.571)] = 0.697$$

$\Rightarrow \boldsymbol{1^{st}\ iteration:}\quad \Rightarrow I_a = 0.697 + \left[\frac{0.697 - 0.709}{3}\right] = 0.693$

$\Rightarrow Error = Exact - Calculated = 0.693 - 0.693 = 0$

$\Rightarrow X_0 + nh = 1 \quad \Rightarrow X_0 = 0 \quad \Rightarrow nh = 1 \quad \Rightarrow n = 8 \quad \therefore h = \frac{1}{8} = 0.125$

X	**0**	**0.125**	**0.25**	**0.375**	**0.5**	**0.625**	**0.75**	**0.875**	**1**
$\boldsymbol{f(x)}$	1	0.889	0.8	0.727	0.667	0.615	0.571	0.533	0.5
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8

$$\Rightarrow I_3 = \int_0^1 \frac{1}{1+x}dx$$

$$= \frac{0.125}{2}[(1+0.5) + 2(0.889 + 0.8 + 0.727 + 0.667 + 0.615 + 0.571 + 0.533)] = 0.694$$

$\Rightarrow \boldsymbol{2^{nd}\ iteration:}\quad \Rightarrow I_b = 0.694 + \left[\frac{0.694 - 0.697}{3}\right] = 0.693$

$\Rightarrow Error = Exact - Calculated = 0.693 - 0.693 = 0$

$\because$ $\boldsymbol{I_a = I_b}$ ***we stop iteration***

❖ ***Find the root of equation*** $\boldsymbol{f(x) = \int_0^2 \sqrt{x}\, dx}$

$$= \int_0^2 x^{0.5}\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = 1.886$$

$\Rightarrow X_0 + nh = 2 \quad \Rightarrow X_0 = 0 \qquad \Rightarrow nh = 2 \qquad \Rightarrow n = 2 \qquad \therefore h = 1$

X	**0**	**1**	**2**
$\boldsymbol{f(x)}$	0	1	1.414
Y	Y_0	Y_1	Y_2

by trapazoidal rule

$$\Rightarrow I_1 = \int_0^2 x^{0.5}\, dx = \frac{1}{2}[(0 + 1.414) + 2(1)] = 1.707$$

$\Rightarrow X_0 + nh = 2 \quad \Rightarrow X_0 = 0 \qquad \Rightarrow nh = 2 \qquad \Rightarrow n = 4 \qquad \therefore h = \frac{2}{4} = 0.5$

X	**0**	**0.5**	**1**	**1.5**	**2**
$\boldsymbol{f(x)}$	0	0.707	1	1.225	1.414
Y	Y_0	Y_1	Y_2	Y_3	Y_4

$$\Rightarrow I_2 = \int_0^2 x^{0.5}\, dx = \frac{0.5}{2}[(0 + 1.414) + 2(0.707 + 1 + 1.225)] = 1.820$$

$\Rightarrow \boldsymbol{1^{st}}$ ***iteration*** : $\Rightarrow I_a = 1.820 + \left[\frac{1.820 - 1.707}{3}\right] = 1.858$

$\Rightarrow Error = Exact - Calculated = 1.886 - 1.858 = 0.028$

$\Rightarrow X_0 + nh = 2 \quad \Rightarrow X_0 = 0 \qquad \Rightarrow nh = 2 \qquad \Rightarrow n = 8 \qquad \therefore h = \frac{2}{8} = 0.25$

X	**0**	**0.25**	**0.5**	**0.75**	**1**	**1.25**	**1.5**	**1.75**	**2**
$\boldsymbol{f(x)}$	0	0.5	0.707	0.866	1	1.118	1.225	1.323	1.414
Y	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8

$$\Rightarrow I_3 = \int_0^2 x^{0.5}\, dx$$

$$= \frac{0.25}{2}[(0 + 1.414) + 2(0.5 + 0.707 + 0.866 + 1 + 1.118 + 1.225 + 1.323)] = 1.862$$

$\Rightarrow \boldsymbol{2^{nd}}$ ***iteration*** : $\Rightarrow I_b = 1.862 + \left[\frac{1.862 - 1.820}{3}\right] = 1.876$

$\Rightarrow Error = Exact - Calculated = 1.886 - 1.876 = 0.01$

- **22.4 : Gauss Quadrature**

$$\Rightarrow \int_a^b f(x)dx \quad = C_1 f(x_1) + C_2 f(x_2)$$

$$\Rightarrow C_1 = C_2 = \frac{b-a}{2} \qquad \Rightarrow x_1 = \frac{b-a}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{b+a}{2} \qquad \Rightarrow x_2 = \frac{b-a}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{b+a}{2}$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_1^2 x\, dx}$

$$= \left[\frac{x^2}{2}\right]_0^1 \quad = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow C_1 = C_2 = \frac{b-a}{2} = \frac{2-1}{2} = \frac{1}{2}$$

$$\Rightarrow x_1 = \frac{b-a}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{b+a}{2} = \frac{1}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{2+1}{2} = 1.211$$

$$\Rightarrow x_2 = \frac{b-a}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{b+a}{2} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{2+1}{2} = 1.789$$

$$\int_1^2 x\, dx \quad = \frac{1}{2} f(1.211) + \frac{1}{2} f(1.789) = \frac{1}{2} \times 1.211 + \frac{1}{2} \times 1.789 = 1.5$$

$$\Rightarrow Error \ = Exact - Calculated = 1.5 - 1.5 = 0$$

❖ $\boldsymbol{Find\ the\ root\ of\ equation\ f(x) = \int_8^{30} 2000\ ln\left[\frac{140000}{140000 - 2100x}\right] - 9.8x\, dx}$

$$\Rightarrow \int_8^{30} f(x)\, dx \ = 11061.335$$

$$\Rightarrow C_1 = C_2 = \frac{b-a}{2} = \frac{30-8}{2} = 11$$

$$\Rightarrow x_1 = \frac{b-a}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{b+a}{2} = 11\left[\frac{-1}{\sqrt{3}}\right] + \frac{30+8}{2} = 12.649$$

$$\Rightarrow x_2 = \frac{b-a}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{b+a}{2} = 11\left[\frac{1}{\sqrt{3}}\right] + \frac{30+8}{2} = 25.351$$

$$\int_8^{30} f(x)\, dx \quad = 11f(12.649) + 11f(25.351) = 11 \times 296.828 + 11 \times 708.487 = 11058.465$$

$$\Rightarrow Error \ = Exact - Calculated = 11061.335 - 11058.465 = 2.870$$

22. 3 Page 651 : $\quad \boldsymbol{f(x) = \int_1^2 \frac{e^x sinx}{1+x^2}\, dx}$

$$\Rightarrow \int_1^2 \frac{e^x sinx}{1 + x^2}\, dx \ = 0.033$$

$$\Rightarrow C_1 = C_2 = \frac{b-a}{2} = \frac{2-1}{2} = \frac{1}{2} = 0.5$$

$$\Rightarrow x_1 = \frac{b-a}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{b+a}{2} = \frac{1}{2}\left[\frac{-1}{\sqrt{3}}\right] + \frac{2+1}{2} = 1.211$$

$$\Rightarrow x_2 = \frac{b-a}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{b+a}{2} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}\right] + \frac{2+1}{2} = 1.789$$

$$\int_1^2 f(x)\,dx \quad = 0.5 \times f(1.211) + 0.5 \times f(1.789) = 0.5 \times 0.026 + 0.5 \times 0.04 = 0.033$$

$$\Rightarrow Error\ = Exact - Calculated = \ 0.033 - 0.033 = 0$$

Chapter 18 : Interpolation and Extrapolation

- **18.1 : Newton's Divided Differences Interpolating Polynomials**
- **Un equal intervals**

$\Rightarrow f(x) = y_0 + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)$
$(x - x_2)f(x_0, x_1, x_2, x_3) + \cdots$

$\Rightarrow f(x) = y_0 + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)$
$(x - x_2)\Delta^3 f(x_0) + \cdots$

$\Rightarrow f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$\Rightarrow f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$ $\qquad \Rightarrow f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\Rightarrow f(x_0, x_1, x_2, x_3) = \dfrac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$

$\Rightarrow f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$ $\qquad \Rightarrow f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$

❖ $\boldsymbol{For\ the\ given\ data\ \ find\ y\ when\ x = 4}$?

X	$x_0 = 0$	$x_1 = 2$	$x_2 = 3$	$x_3 = 5$	$x_4 = 7$
$f(x)$	1	19	20	25	30
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$	$f(x_3) = Y_3$	$f(x_4) = Y_4$

solution :

X	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{19-1}{2-0} = \mathbf{9}$	$\frac{1-9}{3-0} = \mathbf{-2.667}$	$\frac{0.5+2.667}{5-0} = \mathbf{0.633}$	$\frac{-0.1-0.633}{7-0} = \mathbf{-0.105}$
2	19	$\frac{20-19}{3-2} = \mathbf{1}$			
3	20		$\frac{2.5-1}{5-2} = \mathbf{0.5}$		
5	25	$\frac{25-20}{5-3} = \mathbf{2.5}$		$\frac{0-0.5}{7-2} = \mathbf{-0.1}$	
7	30	$\frac{30-25}{7-5} = \mathbf{2.5}$	$\frac{2.5-2.5}{7-3} = \mathbf{0}$		

$\Rightarrow f(x) = 1 + (4 - 0) \times 9 + (4 - 0)(4 - 2) \times (-2.667) + (4 - 0)(4 - 2)(4 - 3) \times 0.633$
$+ (4 - 0)(4 - 2)(4 - 3)(4 - 5) \times (-0.105) = 21.568$

❖ $\boldsymbol{For\ the\ given\ data\ find\ the\ polynomial\ and\ find\ y\ at\ x = 3}$?

X	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 5$
$f(x)$	2	3	12	147
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$	$f(x_3) = Y_3$

solution :

X	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2	$\frac{3-2}{1-0} = \mathbf{1}$	$\frac{9-1}{2-0} = \mathbf{4}$	$\frac{9-4}{5-0} = \mathbf{1}$
1	3	$\frac{12-3}{2-1} = \mathbf{9}$		
2	12		$\frac{45-9}{5-1} = \mathbf{9}$	
5	147	$\frac{147-12}{5-2} = \mathbf{45}$		

$\Rightarrow f(x) = 2 + (x - 0) \times 1 + (x - 0)(x - 1) \times 4 + (x - 0)(x - 1)(x - 2) \times 1$

$= 2 + x + 4(x^2 - x) + (x^3 - 3x^2 + 2x) = \boldsymbol{x^3 + x^2 - x + 2}$

$\Rightarrow f(x) = f(3) = 3^3 + 3^2 - 3 + 2 = 27 + 9 - 3 + 2 = 35$

❖ ***For the data*** $\boldsymbol{Y_0 = -12\,, Y_1 = 0\,, Y_3 = 6\,, Y_4 = 12}$ ***, Find*** $\boldsymbol{Y_2}$? ***and y at*** $\boldsymbol{x = 2}$?

<u>*solution* :</u>

X	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**-12**	$\frac{0+12}{1-0} = \mathbf{12}$	$\frac{3-12}{3-0} = \mathbf{-3}$	$\frac{1+3}{4-0} = \mathbf{1}$
1	**0**	$\frac{6-0}{3-1} = \mathbf{3}$		
3	**6**		$\frac{6-3}{4-1} = \mathbf{1}$	
4	**12**	$\frac{12-6}{4-3} = \mathbf{6}$		

$\Rightarrow f(x) = -12 + (x - 0) \times 12 + (x - 0)(x - 1) \times (-3) + (x - 0)(x - 1)(x - 3) \times 1$

$= -12 + 12x + (-3)(x^2 - x) + (x^3 - 4x^2 + 3x) = \boldsymbol{x^3 - 7x^2 + 18x - 12}$

$\Rightarrow f(x) = f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 8 - 28 + 36 - 12 = 4$

❖ $\boldsymbol{f(x) = \frac{x^2}{1+x^2}}$ $\boldsymbol{, x = 0, 1, 2, 3}$ ***find the polynomial*** ?

<u>*solution* :</u>

X	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**0**	$\frac{0.5-0}{1-0} = \mathbf{0.5}$	$\frac{0.3-0.5}{2-0} = \mathbf{-0.1}$	$\frac{-0.1+0.1}{3-0} = \mathbf{0}$
1	**0.5**	$\frac{0.8-0.5}{2-1} = \mathbf{0.3}$		
2	**0.8**		$\frac{0.1-0.3}{3-1} = \mathbf{-0.1}$	
3	**0.9**	$\frac{0.9-0.8}{3-2} = \mathbf{0.1}$		

$\Rightarrow f(x) = 0 + (x - 0) \times 0.5 + (x - 0)(x - 1) \times (-0.1) + (x - 0)(x - 1)(x - 2) \times 0$

$= 0 + 0.5x + (-0.1)(x^2 - x) + 0 = \boldsymbol{-0.1x^2 + 0.6x}$

- ## 18.2 : Lagrange Interpolating Polynomials

$$\Rightarrow y = f(x) = \left[\frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}\right]y_0 + \left[\frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}\right]y_1 + \left[\frac{(x-x_{n-2})(x-x_{n-1})\dots(x-x_{n+1})}{(x_n-x_{n-2})(x_n-x_{n-1})\dots(x_n-x_{n+1})}\right]y_n$$

❖ ***For the given data find the polynomial then find y at $x = 3$*** ?

X	$x_0 = 0$	$x_1 = 1$	$x_2 = 4$	$x_3 = 5$
$f(x)$	4	3	24	39
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$	$f(x_3) = Y_3$

solution :

$$\Rightarrow y = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right]y_0 + \left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right]y_1 + \left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right]y_2 + \left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right]y_3$$

$$\Rightarrow y = \left[\frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}\right]4 + \left[\frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}\right]3 + \left[\frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}\right]24 + \left[\frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}\right]39$$

$$\Rightarrow y = (x-1)(x-4)(x-5)\left[\frac{4}{-20}\right] + (x-0)(x-4)(x-5)\left[\frac{3}{12}\right] + (x-0)(x-1)(x-5)\left[\frac{24}{-12}\right] + (x-0)(x-1)(x-4)\left[\frac{39}{20}\right]$$

$$\Rightarrow y = [x^3 - 10x^2 + 29x - 20]\left[\frac{4}{-20}\right] + [x^3 - 9x^2 + 20x]\left[\frac{3}{12}\right] + [x^3 - 6x^2 + 5x]\left[\frac{24}{-12}\right] + [x^3 - 5x^2 + 4x]\left[\frac{39}{20}\right]$$

$$\Rightarrow y = \left[\frac{-x^3}{5} + 2x^2 - \frac{29x}{5} + 4\right] + \left[\frac{x^3}{4} - \frac{9x^2}{4} + 5x\right] + [-2x^3 + 12x^2 - 10x] + \left[\frac{39x^3}{20} - \frac{39x^2}{4} + \frac{39x}{5}\right]$$

$$\Rightarrow \boldsymbol{y = f(x) = 2x^2 - 3x + 4}$$

$$\Rightarrow y = \left[\frac{(3-1)(3-4)(3-5)}{(0-1)(0-4)(0-5)}\right]4 + \left[\frac{(3-0)(3-4)(3-5)}{(1-0)(1-4)(1-5)}\right]3 + \left[\frac{(3-0)(3-1)(3-5)}{(4-0)(4-1)(4-5)}\right]24 + \left[\frac{(3-0)(3-1)(3-4)}{(5-0)(5-1)(5-4)}\right]39 = \mathbf{13}$$

- **18.4 : Inverse Interpolation**

$$\Rightarrow x=\left[\frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)}\right]x_0+\left[\frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}\right]x_1$$
$$+\left[\frac{(y-y_{n-2})(y-y_{n-1})\dots(y-y_{n+1})}{(y_n-y)(y_n-y_{n-1})\dots(y_n-y_{n+1})}\right]x_n$$

❖ ***For the given data find x at*** $y=\mathbf{100}$?

X	$x_0=3$	$x_1=5$	$x_2=7$	$x_3=9$	$x_4=11$
$f(x)$	6	24	58	108	174
Y	$f(x_0)=Y_0$	$f(x_1)=Y_1$	$f(x_2)=Y_2$	$f(x_3)=Y_3$	$f(x_4)=Y_4$

solution :

$$\Rightarrow x=\left[\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)}\right]x_0+\left[\frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}\right]x_1$$
$$+\left[\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}\right]x_2+\left[\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}\right]x_3+\left[\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_3)(y_4-y_3)}\right]x_4$$
$$\Rightarrow x=\left[\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}\right]3+\left[\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}\right]5$$
$$+\left[\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}\right]7+\left[\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}\right]9$$
$$+\left[\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}\right]11=\mathbf{8.656}$$

❖ ***For the given data find the polynomial then find x at*** $y=\mathbf{5}$?

X	$x_0=1$	$x_1=3$	$x_2=6$	$x_3=8$
$f(x)$	18	10	-18	90
Y	$f(x_0)=Y_0$	$f(x_1)=Y_1$	$f(x_2)=Y_2$	$f(x_3)=Y_3$

solution :

$$\Rightarrow x=\left[\frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}\right]x_0+\left[\frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}\right]x_1$$
$$+\left[\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}\right]x_2+\left[\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}\right]x_3$$
$$\Rightarrow x=\left[\frac{(y-10)(y+18)(y-90)}{(18-10)(18+18)(18-90)}\right]1+\left[\frac{(y-18)(y+18)(y-90)}{(10-18)(10+18)(10-90)}\right]3$$
$$+\left[\frac{(y-18)(y-10)(y-90)}{(-18-18)(-18-10)(-18-90)}\right]6+\left[\frac{(y-18)(y-10)(y+18)}{(90-18)(90-10)(90+18)}\right]8$$
$$\Rightarrow x=(y-10)(y+18)(y-90)\left[\frac{1}{-20736}\right]+(y-18)(y+18)(y-90)\left[\frac{3}{17920}\right]$$
$$+(y-18)(y-10)(y-90)\left[\frac{6}{-108864}\right]+(y-18)(y-10)(y+18)\left[\frac{8}{622080}\right]$$
$$\Rightarrow x=[y^3-82y^2-900y+16200]\left[\frac{1}{-20736}\right]+[y^3-90y^2-324y+29160]\left[\frac{3}{17920}\right]$$
$$+[y^3-118y^2+2700y-16200]\left[\frac{6}{-108864}\right]+[y^3-10y^2-324y+3240]\left[\frac{8}{622080}\right]$$
$$\Rightarrow \boldsymbol{x=-0.006y^2-0.759y+6.597}$$

Chapter 17 : Least-Square Regression

- **17.1 : Linear Regression**

$\Rightarrow Fitting\ a\ line\ equation\ = y = a_0 + a_1 x$ $\Rightarrow a_0 = \bar{y} - a_1 \bar{x}$

$\Rightarrow a_1 = \dfrac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$ $\Rightarrow \bar{y} = \dfrac{\sum y_i}{n}$ $\Rightarrow \bar{x} = \dfrac{\sum x_i}{n}$

$\Rightarrow Sum\ of\ Squares = S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$

$\Rightarrow Standerd\ derivation\ = S_{\frac{y}{x}} = \sqrt{\dfrac{S_r}{n-2}}$

❖ ***For the given data find***

x	y
1	10
2	35
3	42
4	58

a) ***Fit a linear Regression line*** .
b) ***Error***
c) ***Standard derivation***

solution :

x_i	y_i	$x_i y_i$	x_i^2	$y_i - a_0 - a_1 x_i$	$(y_i - a_0 - a_1 x_i)^2$
1	10	10	1	-3.6	12.96
2	35	70	4	6.3	39.69
3	42	126	9	-1.8	3.24
4	58	232	16	-0.9	0.81
$\sum x_i = 10$	$\sum y_i = 145$	$\sum x_i y_i = 438$	$\sum x_i^2 = 30$	$\sum = 0$	$error = 56.7$

$\Rightarrow n = 4 \quad \Rightarrow a_1 = \dfrac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} = \dfrac{4 \times 438 - 10 \times 145}{4 \times 30 - 10^2} = 15.1$

$\Rightarrow \bar{y} = \dfrac{\sum y_i}{n} = \dfrac{145}{4} = 36.25$

$\Rightarrow \bar{x} = \dfrac{\sum x_i}{n} = \dfrac{10}{4} = 2.5$

$\Rightarrow a_0 = \bar{y} - a_1 \bar{x} = 36.25 - 15.1 \times 2.5 = -1.5$

$\Rightarrow \boldsymbol{linear\ equation\ = y = -1.5 + 15.1x}$

$\Rightarrow S_{\frac{y}{x}} = \sqrt{\frac{S_r}{n-2}} = \sqrt{\frac{56.7}{4-2}} = 5.324$

❖ $\boldsymbol{For\ the\ given\ data\ find}$

x	y
0	5
2	6
4	7
6	6
9	9
11	8
12	7
15	10
17	12
19	12

a) $\boldsymbol{Fit\ a\ linear\ Regression\ line}$.
b) $\boldsymbol{Error}$
c) $\boldsymbol{Standard\ derivation}$

<u>$solution$:</u>

x_i	y_i	$x_i y_i$	${x_i}^2$	$y_i - a_0 - a_1 x_i$	$(y_i - a_0 - a_1 x_i)^2$
0	5	0	0	0.144	0.021
2	6	12	4	0.440	0.194
4	7	28	16	0.736	0.542
6	6	36	36	-0.968	0.937
9	9	81	81	0.976	0.953
11	8	88	121	0.728	0.530
12	7	84	144	-2.080	4.326
15	10	150	225	-0.136	0.018
17	12	204	289	1.160	1.346
19	12	228	361	0.456	0.208
$\sum x_i = 95$	$\sum y_i = 82$	$\sum x_i y_i = 911$	1277	$\sum = 0$	$error = 9.075$

$\Rightarrow n = 10 \quad \Rightarrow a_1 = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum {x_i}^2 - (\sum x_i)^2} = \dfrac{10 \times 911 - 95 \times 82}{10 \times 1277 - 95^2} = 0.351$

$\Rightarrow \bar{y} = \dfrac{\sum y_i}{n} = \dfrac{82}{10} = 8.2$

$\Rightarrow \bar{x} = \dfrac{\sum x_i}{n} = \dfrac{95}{10} = 9.5$

$\Rightarrow a_0 = \bar{y} - a_1 \bar{x} = 8.2 - 0.352 \times 9.5 = 4.856$

$\Rightarrow \boldsymbol{linear\ equation\ =\ y = 4.856 + 0.352\,x}$

$\Rightarrow S_{\frac{y}{x}} = \sqrt{\dfrac{S_r}{n-2}} = \sqrt{\dfrac{9.075}{10-2}} = 1.065$

- ## <u>17.2 : Polynomial Regression</u>

$\Rightarrow y = a_0 + a_1 x + a_2 x^2$

$\Rightarrow \sum y_i = n a_0 + a_1 \sum x_i + a_2 \sum {x_i}^2$

$\Rightarrow \sum x_i y_i = a_0 \sum x_i + a_1 \sum {x_i}^2 + a_2 \sum {x_i}^3$

$\Rightarrow \sum {x_i}^2 y_i = a_0 \sum {x_i}^2 + a_1 \sum {x_i}^3 + a_2 \sum {x_i}^4$

$$\Rightarrow Sum\ of\ Series\ of\ residuals\ or\ error = S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 {x_i}^2)^2$$

$$\Rightarrow Standerd\ derivation = S_{\frac{y}{x}} = \sqrt{\frac{S_r}{n-(m+1)}}$$

$\Rightarrow n = Number\ of\ data$ $\Rightarrow m = degree\ of\ the\ polynomial$

❖ ***For the given data find***

x	y
3	1.6
4	3.6
4	4.4
7	3.4

a) ***Fit a second order Regression line***

b) ***Error***

c) ***Standard derivation***

<u>*solution :*</u>

x_i	y_i	$x_i y_i$	${x_i}^2$	${x_i}^2 y_i$	${x_i}^3$	${x_i}^4$	$(y_i - a_0 - a_1 x_i - a_2 {x_i}^2)^2$
3	1.6	4.8	9	14.4	14.4	81	1.024×10^{-3}
4	3.6	14.4	16	57.6	64	256	4.225×10^{-3}
5	4.4	22	25	110	125	625	4.356×10^{-3}
7	3.4	23.8	49	166.6	343	2401	1.44×10^{-4}
$\sum x_i = 19$	$\sum y_i = 13$	$\sum x_i y_i = 65$	$\sum {x_i}^2 = 99$	$\sum {x_i}^2 y_i = 348.6$	$\sum {x_i}^3 = 559$	$\sum {x_i}^4 = 3363$	$error = 9.749 \times 10^{-3}$

$\Rightarrow \sum y_i = n a_0 + a_1 \sum x_i + a_2 \sum {x_i}^2 = 4a_0 + 19a_1 + 99a_2 = 13$

$\Rightarrow \sum x_i y_i = a_0 \sum x_i + a_1 \sum {x_i}^2 + a_2 \sum {x_i}^3 = 19a_0 + 99a_1 + 559a_2 = 65$

$\Rightarrow \sum {x_i}^2 y_i = a_0 \sum {x_i}^2 + a_1 \sum {x_i}^3 + a_2 \sum {x_i}^4 = 99a_0 + 559a_1 + 3363a_2 = 348.6$

Matrix in Calculator > Mode > 5 : Equ > 2: anX+bnY+cnZ=dn

$$\Rightarrow \begin{bmatrix} 4 & 19 & 99 \\ 19 & 99 & 559 \\ 99 & 559 & 3363 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 65 \\ 348.6 \end{bmatrix} \Rightarrow a_0 = -0.909 \Rightarrow a_1 = 5.305 \Rightarrow a_2 = -0.486$$

$$\Rightarrow \boldsymbol{Second\ order\ equation = y = -0.909 + 5.305x - 0.486x^2}$$

$$\Rightarrow S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 {x_i}^2)^2 = 9.749 \times 10^{-3}$$

$$\Rightarrow S_{\frac{y}{x}} = \sqrt{\frac{S_r}{n-(m+1)}} = \sqrt{\frac{9.749 \times 10^{-3}}{4-(2+1)}} = 0.048$$

Chapter 25 : Runge-Kutta Methods

- ## 25.1 : Euler's Method

$\Rightarrow y_{n+1} = f(x)_{n+1} = y_n + f(x_n, y_n)h \qquad \Rightarrow h = x_n - x_{n-1}$

$\boldsymbol{n = 0} \qquad \Rightarrow y_1 = f(x)_1 = y_0 + f(x_0, y_0)h$

$\boldsymbol{n = 1} \qquad \Rightarrow y_2 = f(x)_2 = y_1 + f(x_1, y_1)h$

❖ $\boldsymbol{Solve \frac{dy}{dx} = x\sqrt{1+y^2} \;\; if\; y(1) = 0\,, find\; y(3)}$?

$\Rightarrow x_0 = 1 \qquad \Rightarrow y_0 = 0 \qquad \because h\ not\ given\ we\ asume\ h\ below\ x_0 \quad \Rightarrow h = 0.4$

X	$x_0 = 1$	$x_1 = 1.4$	$x_2 = 1.8$	$x_3 = 2.2$	$x_4 = 2.6$	$x_5 = 3$
$f(x)$	0	0.4	1.003	2.203	4.009	8.306
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$	$f(x_3) = Y_3$	$f(x_4) = Y_4$	$f(x_5) = Y_5$

$\Rightarrow \boldsymbol{n = 0} \qquad \Rightarrow y_1 = y_0 + f(x_0, y_0)h \quad = 0 + f(1,0) \times 0.4$

$\Rightarrow f(1,0) = x\sqrt{1+y^2} = 1 \times \sqrt{1+0} = 1 \qquad \Rightarrow y_1 = 0 + 1 \times 0.4 = \mathbf{0.4}$

$\Rightarrow \boldsymbol{n = 1} \qquad \Rightarrow y_2 = y_1 + f(x_1, y_1)h \quad = 0.4 + f(1.4\,, 0.4) \times 0.4$

$\Rightarrow f(1.4\,, 0.4) = x\sqrt{1+y^2} = 1.4 \times \sqrt{1+0.4^2} = 1.508$

$\Rightarrow y_2 = 0.4 + 1.508 \times 0.4 = \mathbf{1.003}$

$\Rightarrow \boldsymbol{n = 2} \qquad \Rightarrow y_3 = y_2 + f(x_2, y_2)h \quad = 1.003 + f(1.8\,, 1.003) \times 0.4$

$\Rightarrow f(1.8\,, 1.003) = x\sqrt{1+y^2} = 1.8 \times \sqrt{1+1.003^2} = 2.549$

$\Rightarrow y_3 = 1.003 + 2.549 \times 0.4 = \mathbf{2.203}$

$\Rightarrow \boldsymbol{n = 3} \qquad \Rightarrow y_4 = y_3 + f(x_3, y_3)h \quad = 2.203 + f(2.2\,, 2.203) \times 0.4$

$\Rightarrow f(2.2\,, 2.203) = x\sqrt{1+y^2} = 2.2 \times \sqrt{1+2.203^2} = 4.965$

$\Rightarrow y_4 = 2.203 + 4.965 \times 0.4 = \mathbf{4.009}$

$\Rightarrow \boldsymbol{n = 4} \qquad \Rightarrow y_5 = y_4 + f(x_4, y_4)h \quad = 4.009 + f(2.6\,, 4.009) \times 0.4$

$\Rightarrow f(2.6\,, 4.009) = x\sqrt{1+y^2} = 2.6 \times \sqrt{1+4.009^2} = 10.743$

$\Rightarrow y_5 = 4.009 + 10.743 \times 0.4 = \mathbf{8.306}$

- **Modified Euler's Method**

$\Rightarrow y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, \frac{h}{2}f(x_n, y_n)\right]$

❖ $\boldsymbol{Solve\,\frac{dy}{dx} = y - \frac{2x}{y}\ \ if\ y(0) = 1, find\ y(0.1) and\ y(0.2)}$?

$\Rightarrow x_0 = 0 \qquad \Rightarrow y_0 = 1 \qquad \Rightarrow h = x_1 - x_0 = 0.1 - 0 = 0.1$

X	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
$f(x)$	0	0.805	-0.264
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{n = 0} \qquad \Rightarrow y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, \frac{h}{2}f(x_0, y_0)\right]$

$= 1 + 0.1 \times f\left[0 + \frac{0.1}{2}, \frac{0.1}{2}f(0,1)\right]$

$\Rightarrow f(0,1) = y - \frac{2x}{y} = 1 - \frac{2\times 0}{1} = 1 - 0 = 1$

$= 1 + 0.1 \times f[0.05, 0.05 \times 1] = 1 + 0.1f[0.05, 0.05]$

$\Rightarrow f(0.05, 0.05) = y - \frac{2x}{y} = 0.05 - \frac{2\times 0.05}{0.05} = 0.05 - 2 = -1.95$

$\Rightarrow y_1 = 1 + 0.1 \times (-1.95) = \mathbf{0.805}$

$\Rightarrow \boldsymbol{n = 1} \qquad \Rightarrow y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, \frac{h}{2}f(x_1, y_1)\right]$

$= 0.805 + 0.1 \times f\left[0.1 + \frac{0.1}{2}, \frac{0.1}{2}f(0.1, 0.805)\right]$

$\Rightarrow f(0.1, 0.805) = y - \frac{2x}{y} = 0.805 - \frac{2\times 0.1}{0.805} = 0.805 - 0.248 = 0.557$

$= 0.805 + 0.1 \times f[0.15, 0.05 \times 0.557] = 0.805 + 0.1f[0.15, 0.028]$

$\Rightarrow f(0.15, 0.028) = y - \frac{2x}{y} = 0.028 - \frac{2\times 0.15}{0.028} = 0.028 - 10.714 = -10.686$

$\Rightarrow y_2 = 0.805 + 0.1 \times (-10.686\ ) = -\mathbf{0.264}$

❖ $\boldsymbol{Solve\,\frac{dy}{dx} = \frac{x - y}{x + y}\ \ if\ y(2) = 1, find\ y(4) and\ take\ h = 1}$

X	$x_0 = 2$	$x_1 = 3$	$x_2 = 4$
$f(x)$	1	1.875	2.811
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{n = 0} \qquad \Rightarrow y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, \frac{h}{2}f(x_0, y_0)\right]$

$= 1 + 1 \times f\left[2 + \frac{1}{2}, \frac{1}{2}f(2, 1)\right]$

$\Rightarrow f(2, 1) = \frac{x-y}{x+y} = \frac{2-1}{2+1} = \frac{1}{3} = 0.333$

$= 1 + f[2.5, 0.5 \times 0.333] = 1 + f[2.5, 0.167]$

$\Rightarrow f(2.5, 0.167) = \frac{x-y}{x+y} = \frac{2.5-0.167}{2.5+0.167} = 0.875$

$\Rightarrow y_1 = 1 + 0.875 = \mathbf{1.875}$

$\Rightarrow \boldsymbol{n = 1} \qquad \Rightarrow y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, \frac{h}{2}f(x_1, y_1)\right]$

$= 1.875 + 1 \times f\left[3 + \frac{1}{2}, \frac{1}{2}f(3, 1.875)\right]$

$\Rightarrow f(3, 1.875) = \frac{x-y}{x+y} = \frac{3-1.875}{3+1.875} = 0.231$

$= 1 + f[3.5, 0.5 \times 0.231] = 1 + f[3.5, 0.115]$

$\Rightarrow f(3.5, 0.115) = \frac{x-y}{x+y} = \frac{3.5-0.115}{3.5+0.115} = 0.936$

$\Rightarrow y_2 = 1.875 + 0.936 = \mathbf{2.811}$

- ## 25.2 : Improvement of Euler's Method

$\Rightarrow y_{m+1} = y_m + \frac{h}{2}\left[f(x_m, y_m) + f[x_m + h\,, y_m + hf(x_m, y_m)]\right]$

❖ $\boldsymbol{Solve \frac{dy}{dx} = -x^2 - y = 4 \;\; if\ y(0) = 1\,, find\ y(0.2)\ take\ h = 0.1}$?

X	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
$f(x)$	1	1.526	2.109
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{m = 0} \qquad \Rightarrow y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f[x_0 + h\,, y_0 + hf(x_0, y_0)]\right]$

$= 1 + \frac{0.1}{2}\left[f(0{,}1) + f[0 + 0.1\,, 1 + 0.1 \times f(0{,}1)]\right]$

$\Rightarrow f(0\,, 1) = 4 + x^2 + y = 4 + 0 + 1 = 5$

$= 1 + 0.05\left[5 + f[0.1\,, 1 + 0.1 \times 5]\right]$

$= 1 + 0.05[5 + f(0.1\,, 1.5)]$

$\Rightarrow f(0.1\,, 1.5) = 4 + x^2 + y = 4 + 0.1^2 + 1.5 = 5.510$

$\Rightarrow y_1 = 1 + 0.05[5 + 5.510] = \boldsymbol{1.526}$

$\Rightarrow \boldsymbol{m = 1} \qquad \Rightarrow y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f[x_1 + h\,, y_1 + hf(x_1, y_1)]\right]$

$= 1.526 + \frac{0.1}{2}\left[f(0.1\,, 1.526) + f[0.1 + 0.1\,, 1.526 + 0.1 \times f(0.1\,,1.526)]\right]$

$\Rightarrow f(0.1\,, 1.526) = 4 + 0.1^2 + 1.526 = 5.536$

$= 1.526 + 0.05\left[5.536 + f[0.2\,, 1.526 + 0.1 \times 5.536]\right]$

$= 1.526 + 0.05[5.536 + f(0.2\,, 2.079)]$

$\Rightarrow f(0.2\,, 2.079) = 4 + x^2 + y = 4 + 0.2^2 + 2.079 = 6.120$

$\Rightarrow y_2 = 1.526 + 0.05[5.536 + 6.120] = \boldsymbol{2.109}$

- **25.3 : Runge-Kutta Method**
- **Second order**

$\Rightarrow y_{n+1} = y_n + \Delta y_n$

$\Rightarrow \Delta y_n = \frac{1}{6}[k_1 + 4k_2 + k_3]$

$\Rightarrow k_1 = hf(x_n , y_n)$

$\Rightarrow k_2 = hf\left(x_n + \frac{h}{2} , y_n + \frac{k_1}{2}\right)$

$\Rightarrow k_3 = hf(x_n + h , y_n + 2k_2 - k_1)$

❖ $\boldsymbol{Solve \frac{dy}{dx} = y - x \;\; if\; y(0) = 2 , find\; y(0.3)\; take\; h = 0.1}$?

X	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_2 = 0.3$
$f(x)$	2	2.205	2.421	2.109
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{n = 0}$

$\Rightarrow k_1 = hf(x_0 , y_0) = 0.1 \times f(0 , 2) = 0.1 \times (2 - 0) = \mathbf{0.2}$

$\Rightarrow k_2 = hf\left[x_0 + \frac{h}{2} , y_0 + \frac{k_1}{2}\right] = 0.1 \times f\left[0 + \frac{0.1}{2} ,2 + \frac{0.2}{2}\right] = 0.1 \times f(0.05 ,2.1)$

$\Rightarrow f(0.05 ,2.1) = 2.1 - 0.05 = 2.05 \;\; \Rightarrow k_2 = 0.1 \times 2.05 = \mathbf{0.205}$

$\Rightarrow k_3 = hf[x_0 + h , y_0 + 2k_2 - k_1] = 0.1 \times f[0 + 0.1 ,2 + 2(0.205) - 0.2] = 0.1 \times f(0.1 ,2.21)$

$\Rightarrow f(0.1 ,2.21) = 2.21 - 0.1 = 2.11 \;\; \Rightarrow k_3 = 0.1 \times 2.11 = \mathbf{0.211}$

$\Rightarrow \Delta y_0 = \frac{1}{6}[0.2 + 4(0.205) + 0.211] = \mathbf{0.205}$

$\Rightarrow y_1 = y_0 + \Delta y_0 = 2 + 0.205 = \mathbf{2.205}$

$\Rightarrow \boldsymbol{n = 1}$

$\Rightarrow k_1 = hf(x_1 , y_1) = 0.1 \times f(0.1 , 2.205) = 0.1 \times (2.205 - 0.1) = \mathbf{0.211}$

$\Rightarrow k_2 = hf\left[x_1 + \frac{h}{2} , y_1 + \frac{k_1}{2}\right] = 0.1 \times f\left[0.1 + \frac{0.1}{2} ,2.205 + \frac{0.211}{2}\right] = 0.1 \times f(0.15 ,2.311)$

$\Rightarrow f(0.15 ,2.311) = 2.311 - 0.15 = 2.161 \;\; \Rightarrow k_2 = 0.1 \times 2.161 = \mathbf{0.216}$

$\Rightarrow k_3 = hf[x_1 + h , y_1 + 2k_2 - k_1]$

$= 0.1 \times f[0.1 + 0.1 , 2.205 + 2(0.216) - 0.211] = 0.1 \times f(0.1 ,2.426)$

$\Rightarrow f(0.2 ,2.426) = 2.426 - 0.1 = 2.226 \;\; \Rightarrow k_3 = 0.1 \times 2.226 = \mathbf{0.223}$

$\Rightarrow \Delta y_1 = \frac{1}{6}[0.211 + 4(0.216) + 0.223] = \mathbf{0.216}$

$\Rightarrow y_2 = y_1 + \Delta y_1 = 2.205 + 0.216 = \mathbf{2.421}$

$\Rightarrow \boldsymbol{n = 2}$

$\Rightarrow k_1 = hf(x_2 , y_2) = 0.1 \times f(0.2 , 2.421) = 0.1 \times (2.421 - 0.2) = \mathbf{0.222}$

$\Rightarrow k_2 = hf\left[x_2 + \frac{h}{2} , y_2 + \frac{k_1}{2}\right] = 0.1 \times f\left[0.2 + \frac{0.1}{2} ,2.421 + \frac{0.222}{2}\right] = 0.1 \times f(0.25 ,2.532)$

$\Rightarrow f(0.25 ,2.532) = 2.532 - 0.25 = 2.332 \;\; \Rightarrow k_2 = 0.1 \times 2.332 = \mathbf{0.233}$

$\Rightarrow k_3 = hf[x_2 + h , y_2 + 2k_2 - k_1]$

$= 0.1 \times f[0.2 + 0.1 , 2.421 + 2(0.233) - 0.222] = 0.1 \times f(0.3 ,2.665)$

$\Rightarrow f(0.3 ,2.665) = 2.665 - 0.3 = 2.365 \;\; \Rightarrow k_3 = 0.1 \times 2.365 = \mathbf{0.237}$

$\Rightarrow \Delta y_2 = \frac{1}{6}[0.222 + 4(0.233) + 0.237] = \mathbf{0.232}$

$\Rightarrow y_3 = y_2 + \Delta y_2 = 2.421 + 0.232 = \mathbf{2.653}$

- <u>Fourth order</u>

  $\Rightarrow y_{n+1} = y_n + \Delta y_n$

  $\Rightarrow \Delta y_n = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

  $\Rightarrow k_1 = hf(x_n\,, y_n)$

  $\Rightarrow k_2 = hf\left(x_n + \frac{h}{2}\,, y_n + \frac{k_1}{2}\right)$

  $\Rightarrow k_3 = hf\left(x_n + \frac{h}{2}\,, y_n + \frac{k_2}{2}\right)$

  $\Rightarrow k_4 = hf(x_n + h\,, y_n + k_3)$

  ❖ $\boldsymbol{Solve\,\frac{dy}{dx} = y - x \;\; if\; y(0) = 2\,, find\; y(0.2)\; take\; h = 0.1}$?

X	$\boldsymbol{x_0 = 0}$	$\boldsymbol{x_1 = 0.1}$	$\boldsymbol{x_2 = 0.2}$
$\boldsymbol{f(x)}$	2	2.205	2.421
Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{n = 0}$

$\Rightarrow k_1 = hf(x_0\,, y_0) = 0.1 \times f(0\,,2) = 0.1 \times (2 - 0) = \boldsymbol{0.2}$

$\Rightarrow k_2 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right] = 0.1 \times f\left[0 + \frac{0.1}{2}\,,2 + \frac{0.2}{2}\right] = 0.1 \times f(0.05\,,2.1)$

$\Rightarrow f(0.05\,,2.1) = 2.1 - 0.05 = 2.05 \;\; \Rightarrow k_2 = 0.1 \times 2.05 = \boldsymbol{0.205}$

$\Rightarrow k_3 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right] = 0.1 \times f\left[0 + \frac{0.1}{2}\,,2 + \frac{0.205}{2}\right] = 0.1 \times f(0.05\,,2.103)$

$\Rightarrow f(0.05\,,2.103) = 2.103 - 0.05 = 2.053 \;\; \Rightarrow k_3 = 0.1 \times 2.053 = \boldsymbol{0.205}$

$\Rightarrow k_4 = hf(x_0 + h\,, y_0 + k_3) = 0.1 \times f(0 + 0.1\,,2 + 0.205) = 0.1 \times f(0.1\,, 2.205)$

$\Rightarrow f(0.1\,, 2.205) = 2.205 - 0.1 = 2.105 \;\; \Rightarrow k_4 = 0.1 \times 2.105 = \boldsymbol{0.211}$

$\Rightarrow \Delta y_0 = \frac{1}{6}[0.2 + 2(0.205) + 2(0.205) + 0.211] = \boldsymbol{0.205}$

$\Rightarrow y_1 = y_0 + \Delta y_0 = 2 + 0.205 = \boldsymbol{2.205}$

$\Rightarrow \boldsymbol{n = 1}$

$\Rightarrow k_1 = hf(x_1\,, y_1) = 0.1 \times f(0.1\,, 2.205) = 0.1 \times (2.205 - 0.1) = \boldsymbol{0.211}$

$\Rightarrow k_2 = hf\left[x_1 + \frac{h}{2}\,, y_1 + \frac{k_1}{2}\right] = 0.1 \times f\left[0.1 + \frac{0.1}{2}\,,2.205 + \frac{0.211}{2}\right] = 0.1 \times f(0.15\,,2.311)$

$\Rightarrow f(0.15\,,2.311) = 2.311 - 0.15 = 2.161 \;\; \Rightarrow k_2 = 0.1 \times 2.161 = \boldsymbol{0.216}$

$\Rightarrow k_3 = hf\left[x_1 + \frac{h}{2}\,, y_1 + \frac{k_2}{2}\right] = 0.1 \times f\left[0.1 + \frac{0.1}{2}\,,2.205 + \frac{0.216}{2}\right] = 0.1 \times f(0.15\,,2.313)$

$\Rightarrow f(0.15\,,2.313) = 2.313 - 0.15 = 2.163 \;\; \Rightarrow k_3 = 0.1 \times 2.163 = \boldsymbol{0.216}$

$\Rightarrow k_4 = hf(x_1 + h\,, y_1 + k_3) = 0.1 \times f(0.1 + 0.1\,,2.205 + 0.216) = 0.1 \times f(0.2\,, 2.421)$

$\Rightarrow f(0.2\,, 2.421) = 2.421 - 0.2 = 2.221 \;\; \Rightarrow k_4 = 0.1 \times 2.221 = \boldsymbol{0.222}$

$\Rightarrow \Delta y_1 = \frac{1}{6}[0.211 + 2(0.216) + 2(0.216) + 0.222] = \boldsymbol{0.216}$

$\Rightarrow y_2 = y_1 + \Delta y_1 = 2.205 + 0.216 = \boldsymbol{2.421}$

  ❖ $\boldsymbol{Solve\,\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x} \;\; if\; x_0 = 1, y_0 = 0\; find\; at\; x = 1.2\; and\; 1.4 \; = 0.1}$?

X	$\boldsymbol{x_0 = 1}$	$\boldsymbol{x_1 = 1.2}$	$\boldsymbol{x_2 = 1.4}$
$\boldsymbol{f(x)}$	0	0.141	2.421

Y	$f(x_0) = Y_0$	$f(x_1) = Y_1$	$f(x_2) = Y_2$

$\Rightarrow \boldsymbol{n = 0}$

$$\Rightarrow k_1 = hf(x_0\,, y_0) = 0.2 \times f(1\,, 0) = 0.2 \times \left[\frac{0 + e^1}{1^2 + e^1}\right] = 0.2 \times 0.731 = \mathbf{0.146}$$

$$\Rightarrow k_2 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right] = 0.2 \times f\left[1 + \frac{0.2}{2}\,, 0 + \frac{0.146}{2}\right] = 0.2 \times f(1.1\,, 0.073)$$

$$\Rightarrow f(1.1\,, 0.073) = \left[\frac{2 \times 1.1 \times 0.037 + e^{1.1}}{1.1^2 + 1.1 \times e^{1.1}}\right] = 0.683 \quad \Rightarrow k_2 = 0.2 \times 0.683 = \mathbf{0.137}$$

$$\Rightarrow k_3 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right] = 0.2 \times f\left[1 + \frac{0.2}{2}\,, 0 + \frac{0.137}{2}\right] = 0.2 \times f(1.1\,, 0.069)$$

$$\Rightarrow f(1.1\,, 0.069) = \left[\frac{2 \times 1.1 \times 0.069 + e^{1.1}}{1.1^2 + 1.1 \times e^{1.1}}\right] = 0.699 \quad \Rightarrow k_2 = 0.2 \times 0.699 = \mathbf{0.140}$$

$$\Rightarrow k_4 = hf(x_0 + h\,, y_0 + k_3) = 0.2 \times f(1 + 0.2\,, 0 + 0.140) = 0.1 \times f(1.2\,, 0.140)$$

$$\Rightarrow f(1.2\,, 0.140) = \left[\frac{2 \times 1.2 \times 0.14 + e^{1.2}}{1.2^2 + 1.2 \times e^{1.2}}\right] = 0.674 \quad \Rightarrow k_2 = 0.2 \times 0.674 = \mathbf{0.135}$$

$$\Rightarrow \Delta y_0 = \frac{1}{6}[0.146 + 2(0.137) + 2(0.140) + 0.135] = \mathbf{0.139}$$

$$\Rightarrow y_1 = y_0 + \Delta y_0 = 0 + 0.139 = \mathbf{0.139}$$

$\Rightarrow \boldsymbol{n = 1}$

$$\Rightarrow k_1 = hf(x_1\,, y_1) = 0.2 \times f(1.2\,, 0.139) = 0.2 \times \left[\frac{2 \times 1.2 \times 0.139 + e^{1.2}}{1.2^2 + 1.2 \times e^{1.2}}\right] = 0.2 \times 0.674 = \mathbf{0.135}$$

$$\Rightarrow k_2 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right] = 0.2 \times f\left[1.2 + \frac{0.2}{2}\,, 0.139 + \frac{0.135}{2}\right] = 0.2 \times f(1.3\,, 0.207)$$

$$\Rightarrow f(1.3\,, 0.207) = \left[\frac{2 \times 1.2 \times 0.207 + e^{1.2}}{1.1^2 + 1.2 \times e^{1.2}}\right] = 0.735 \quad \Rightarrow k_2 = 0.2 \times 0.735 = \mathbf{0.147}$$

$$\Rightarrow k_3 = hf\left[x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right] = 0.2 \times f\left[1.2 + \frac{0.2}{2}\,, 0.139 + \frac{0.147}{2}\right] = 0.2 \times f(1.3\,, 0.213)$$

$$\Rightarrow f(1.3\,, 0.213) = \left[\frac{2 \times 1.1 \times 0.069 + e^{1.1}}{1.1^2 + 1.1 \times e^{1.1}}\right] = 0.699 \quad \Rightarrow k_3 = 0.2 \times 0.699 = \mathbf{0.140}$$

$$\Rightarrow k_4 = hf(x_0 + h\,, y_0 + k_3) = 0.2 \times f(1 + 0.2\,, 0 + 0.140) = 0.1 \times f(1.2\,, 0.140)$$

$$\Rightarrow f(1.2\,, 0.140) = \left[\frac{2 \times 1.2 \times 0.14 + e^{1.2}}{1.2^2 + 1.2 \times e^{1.2}}\right] = 0.674 \quad \Rightarrow k_4 = 0.2 \times 0.674 = \mathbf{0.135}$$

$$\Rightarrow \Delta y_1 = \frac{1}{6}[0.146 + 2(0.137) + 2(0.140) + 0.135] = \mathbf{0.141}$$

$$\Rightarrow y_2 = y_1 + \Delta y_1 = 0 + 0.141 = \mathbf{0.141}$$

- ## 25.3 : System of Equations

$\Rightarrow \frac{dy}{dx} = f(x, y, z)$
$\Rightarrow y_{n+1} = y_n + \Delta y_n$
$\Rightarrow \Delta y_n = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
$\Rightarrow k_1 = hf(x_n, y_n, z_n)$
$\Rightarrow k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right)$
$\Rightarrow k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right)$
$\Rightarrow k_4 = hf(x_n + h, y_n + k_3, z_n + l_3)$

$\Rightarrow \frac{dz}{dx} = g(x, y, z)$
$\Rightarrow y_{n+1} = z_n + \Delta z_n$
$\Rightarrow \Delta y_n = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$\Rightarrow l = hg(x_n, y_n, z_n)$
$\Rightarrow l_2 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right)$
$\Rightarrow l_3 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right)$
$\Rightarrow l_4 = hf(x_n + h, y_n + k_3, z_n + l_3)$

❖ $\boldsymbol{Solve \frac{dy}{dx} = x + z \quad and \quad \frac{dz}{dx} = x - y \quad if\ y = 0, z = 1, x = 0}$
$\boldsymbol{find\ y(0.2)\ take\ h = 0.1}$?

X	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
Y	$y_0 = 0$	0.1048	0.218
Z	$z_0 = 1$	1	0.9986

$\Rightarrow \boldsymbol{n = 0}$

$\Rightarrow \frac{dy}{dx} = f(x, y, z)$
$\Rightarrow k_1 = hf(x_0, y_0, z_0) = 0.1 \times f(0, 0, 1)$
$\Rightarrow f(0, 0, 1) = x + z = 0 + 1 = 1$
$\Rightarrow k_1 = 0.1 \times 1 = \boldsymbol{0.1}$
$\Rightarrow k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.1 \times f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}, 1 + 0\right)$
$\Rightarrow f(0.05, 0.05, 1) = 0.05 + 1 = 1.05$
$\Rightarrow k_2 = 0.1 \times 1.05 = \boldsymbol{0.105}$
$\Rightarrow k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.1 \times f\left(0 + \frac{0.1}{2}, 0 + \frac{0.105}{2}, 1 + 0\right)$
$\Rightarrow f(0.05, 0.053, 1) = 0.05 + 1 = 1.05$
$\Rightarrow k_3 = 0.1 \times 1.05 = \boldsymbol{0.105}$
$\Rightarrow k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.1 \times f(0 + 0.1, 0 + 0.105, 1 - 0.0003)$
$\Rightarrow f(0.1, 0.105, 0.9997)$
$= 0.1 + 0.9997 = 1.0997$
$\Rightarrow k_4 = 0.1 \times 1.0997 = \boldsymbol{0.10997}$
$\Rightarrow \Delta y_0 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
$= \frac{1}{6}[0.1 + 2(0.105) + 2(0.105) + 0.10997]$
$= \boldsymbol{0.1048}$
$\Rightarrow y_1 = y_0 + \Delta y_0$
$= 0 + 0.1048 = \boldsymbol{0.1048}$

$\Rightarrow \frac{dz}{dx} = g(x, y, z)$
$\Rightarrow l_1 = hg(x_0, y_0, z_0) = 0.1 \times g(0, 0, 1)$
$\Rightarrow g(0, 0, 1) = x - y = 0 - 0 = 0$
$\Rightarrow l_1 = 0.1 \times 0 = \boldsymbol{zero}$
$\Rightarrow l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.1 \times g\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}, 1 + 0\right)$
$\Rightarrow g(0.05, 0.05, 1) = 0.05 - 0.05 = 0$
$\Rightarrow l_2 = 0.1 \times 0 = \boldsymbol{zero}$
$\Rightarrow l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.1 \times g\left(0 + \frac{0.1}{2}, 0 + \frac{0.105}{2}, 1 + 0\right)$
$\Rightarrow g(0.05, 0.053, 1)$
$= 0.05 - 0.053 = -0.003$
$\Rightarrow l_3 = 0.1 \times -0.003 = \boldsymbol{-3 \times 10^{-4}}$
$\Rightarrow l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$
$\Rightarrow g(0.1, 0.105, 0.9997)$
$= 0.1 - 0.105 = -0.005$
$\Rightarrow k_4 = 0.1 \times -0.005 = \boldsymbol{-5 \times 10^{-4}}$
$\Rightarrow \Delta z_0 = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$= \frac{1}{6}[0 + 0 + 2(-3 \times 10^{-4}) - 5 \times 10^{-4}]$
$= \boldsymbol{-1.833 \times 10^{-4}}$
$\Rightarrow z_1 = z_0 + \Delta z_0$
$= 1 - 1.833 \times 10^{-4} \cong \boldsymbol{1}$

$\Rightarrow \boldsymbol{n = 1}$

$\Rightarrow k_1 = hf(x_1 , y_1 , z_1)$
$= 0.1 \times f(0.1 , 0.1048 , 1)$
$\Rightarrow f(0.1 , 0.1048 , 1) = 0.1 + 1 = 1.1$
$\Rightarrow k_1 = 0.1 \times 1.1 = \mathbf{0.11}$
$\Rightarrow k_2 = hf\left(x_1 + \frac{h}{2} , y_1 + \frac{k_1}{2} , z_1 + \frac{l_1}{2}\right)$
$= 0.1 \times f\left(\begin{matrix} 0.1 + \frac{0.1}{2} , 0.1048 + \frac{0.11}{2} , \\ 1 + \frac{-0.00048}{2} \end{matrix}\right)$
$\Rightarrow f(0.15 , 0.1598 , 0.99976)$
$= 0.15 + 0.99976 = 1.1498$
$\Rightarrow k_2 = 0.1 \times 1.1498 = \mathbf{0.11498}$
$\Rightarrow k_3 = hf\left(x_1 + \frac{h}{2} , y_1 + \frac{k_2}{2} , z_1 + \frac{l_2}{2}\right)$
$= 0.1 \times f(0.15 + 0.162288 , 1.00049)$
$\Rightarrow f(0.15 + 0.162288 , 1.00049)$
$= 0.15 + 1.00049 = 1.15049$
$\Rightarrow k_3 = 0.1 \times 1.15049 = \mathbf{0.115049}$
$\Rightarrow k_4 = hf(x_1 + h , y_1 + k_3 , z_1 + l_3)$
$= 0.1 \times f(0.2 , 0.21985 , 0.9987)$
$\Rightarrow f(0.2 , 0.21985 , 0.9987)$
$= 0.2 + 0.9987 = 1.1987$
$\Rightarrow k_4 = 0.1 \times 1.1987 = \mathbf{0.11987}$
$\Rightarrow \Delta y_1 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
$= \frac{1}{6}\left[\begin{matrix} 0.1 + 2(0.11498) + 2(0.115049) \\ +0.11987 \end{matrix}\right]$
$= \mathbf{0.11332}$
$\Rightarrow y_2 = y_1 + \Delta y_1$
$= 0.1048 + 0.11332 = \mathbf{0.218}$

$\Rightarrow l_1 = hg(x_1 , y_1 , z_1)$
$\Rightarrow g(0.1 , 0.1048 , 1)$
$= 0.1 - 0.1048 = -0.0048$
$\Rightarrow l_1 = 0.1 \times -0.0048 = -\mathbf{0.00048}$
$\Rightarrow l_2 = hg\left(x_1 + \frac{h}{2} , y_1 + \frac{k_1}{2} , z_1 + \frac{l_1}{2}\right)$
$\Rightarrow g(0.15 , 0.1598 , 0.99976)$
$= 0.15 - 0.1598 = -0.0098$
$\Rightarrow l_2 = 0.1 \times -0.0098 = -\mathbf{0.00098}$
$\Rightarrow l_3 = hg\left(x_1 + \frac{h}{2} , y_1 + \frac{k_2}{2} , z_1 + \frac{l_2}{2}\right)$
$= 0.1 \times g(0.15 + 0.162288 , 1.00049)$
$\Rightarrow g(0.15 + 0.162288 , 1.00049)$
$= 0.15 - 0.162288 = -0.012288$
$\Rightarrow l_3 = 0.1 \times -0.012288 = -\mathbf{0.0012288}$
$\Rightarrow l_4 = hg(x_1 + h , y_1 + k_3 , z_1 + l_3)$
$= 0.1 \times g(0.2 , 0.219849 , 0.9987)$
$\Rightarrow g(0.2 , 0.219849 , 0.9987)$
$= 0.2 - 0.1219849 = 0.078$
$\Rightarrow l_4 = 0.1 \times 0.0377 = -\mathbf{0.00377}$
$\Rightarrow \Delta z_0 = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$= \frac{1}{6}\left[\begin{matrix} -0.00048 + 2(-0.00098) + \\ 2(-0.0012288) - 0.00377 \end{matrix}\right]$
$= -\mathbf{0.0014}$
$\Rightarrow z_2 = z_1 + \Delta z_1$
$= 1 - 0.0014 \cong \mathbf{0.9986}$

- <u>Gauss Elimination</u>
  - ❖ ***Solve***

  $\Rightarrow \boldsymbol{x_1 - x_2 + x_3 = 1}$ $\quad \Rightarrow \boldsymbol{-3x_1 + 2x_2 - 3x_3 = -6}$ $\quad \Rightarrow \boldsymbol{2x_1 - 5x_2 + 4x_3 = 5}$

<u>*solution :*</u>

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} \qquad \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow Ax = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix}$$

$\Rightarrow (A|B) = arangment\ matrix$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right) \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right) \begin{array}{l} \rightarrow R_1 \rightarrow R_1 \\ \rightarrow R_2 \rightarrow 3R_1 + R_2 \\ \rightarrow R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$\Rightarrow \boldsymbol{3R_1 + R_2}$

3	−3	3	3
−3	2	−3	−6
0	−1	0	−3

$\Rightarrow \boldsymbol{R_3 - 2R_1}$

2	−5	4	5
2	−2	2	2
0	−3	2	3

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{array}\right) \begin{array}{l} \rightarrow R_1 \rightarrow R_1 \\ \rightarrow R_2 \rightarrow R_2 \\ \rightarrow R_3 \rightarrow R_3 - 3R_2 \end{array}$$

$\Rightarrow \boldsymbol{R_3 - 3R_2}$

0	−3	2	3
0	−3	0	−9
0	0	2	12

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 + x_3 = 1$ $\qquad \Rightarrow \boldsymbol{x_1 = -2}$

$\Rightarrow 0 - x_2 + 0 = -3$ $\qquad \Rightarrow \boldsymbol{x_2 = 3}$

$\Rightarrow 0 - 0 + 2x_3 = 12$ $\qquad \Rightarrow \boldsymbol{x_3 = 6}$

- **Gauss Jordan**
  - ***Solve***

  $\Rightarrow x_1 - x_2 + x_3 = 1 \quad \Rightarrow -3x_1 + 2x_2 - 3x_3 = -6 \quad \Rightarrow 2x_1 - 5x_2 + 4x_3 = 5$

solution :

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} \qquad \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow Ax = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix}$$

$\Rightarrow (A|B) = arangment\ matrix$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right) \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right) \begin{array}{l} \to R_1 \to R_1 \\ \to R_2 \to 3R_1 + R_2 \\ \to R_3 \to R_3 - 2R_1 \end{array}$$

$\Rightarrow 3R_1 + R_2$

3	−3	3	3
−3	2	−3	−6
0	−1	0	−3

$\Rightarrow R_3 - 2R_1$

2	−5	4	5
2	−2	−2	−2
0	−3	2	3

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{array}\right) \begin{array}{l} \to R_1 \to R_1 \\ \to R_2 \to R_2 \\ \to R_3 \to R_3 - 3R_2 \end{array}$$

$\Rightarrow R_3 - 2R_1$

0	−3	2	3
0	−3	0	−9
0	0	2	12

$$\left(\begin{array}{ccc|c} 2 & -2 & 1 & -10 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{array}\right) \begin{array}{l} \to R_1 \to 2R_1 - R_3 \\ \to R_2 \to R_2 \\ \to R_3 \to R_3 \end{array}$$

$\Rightarrow 2R_1 - R_3$

2	−2	2	2
0	0	−2	−12
2	−2	0	−10

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{array}\right) \begin{array}{l} \to R_1 \to R_1 - 2R_2 \\ \to R_2 \to R_2 \\ \to R_3 \to R_3 \end{array}$$

$\Rightarrow R_1 - 2R_2$

2	−2	0	−10
0	−2	0	−6
2	0	0	−4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} \Rightarrow \frac{R_1}{2} \Rightarrow \frac{R_2}{-1} \Rightarrow \frac{R_3}{2}$$

$\Rightarrow x_1 = -2$

$\Rightarrow x_2 = 3$

$\Rightarrow x_3 = 6$