CHAPTER

5

Exponential and Logarithmic Functions

MOST of the functions we have considered so far have been polynomial or rational functions, with a few others involving roots of polynomial or rational functions. Functions that can be expressed in terms of addition, subtraction, multiplication, division, and the taking of roots of variables and constants are called *algebraic functions.*

In Chapter 5 we introduce and investigate the properties of *exponential functions* and *logarithmic functions.* These functions are not algebraic; they belong to the class of *transcendental functions.* Exponential and logarithmic functions are used to model a variety of real-world phenomena: growth of populations of people, animals, and bacteria; radioactive decay; epidemics; absorption of light as it passes through air, water, or glass; magnitudes of sounds and earthquakes. We consider applications in these areas plus many more in the sections that follow.

SECTIONS

5-1 Exponential Functions

In Section 5-1 we introduce exponential functions and investigate their properties and graphs. We also study applications of exponential functions in the mathematics of finance.

Exponential Functions

Let's start by noting that the functions f and g given by

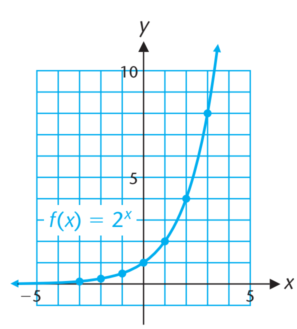
$$f(x) = 2^x \qquad \text{and} \qquad g(x) = x^2$$

are not the same function. Whether a variable appears as an exponent with a constant base or as a base with a constant exponent makes a big difference. The function g is a quadratic function. We discussed quadratic functions in Chapter 3. The function f is a new type of function called an *exponential function.*

The values of the exponential function $f(x) = 2^x$ for x an integer are easy to compute. So if you were asked to graph f, you would probably construct a table of values, plot points, and join those points with a smooth curve (see Fig. 1).

Figure 1 $f(x) = 2^x$.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



One might raise the objection that we have not defined 2^x for each real number x. It is true that if $x = \frac{m}{n}$ is a rational number, then $2^x = \sqrt[n]{2^m}$ (see Section R-2). But what does

$$2^{\sqrt{2}}$$

mean? The question is not easy to answer at this time. In fact, a precise definition of $2^{\sqrt{2}}$ must wait for more advanced courses, where we can show that, if b is a positive real number and x is any real number, then

$$b^x$$

names a real number, and the graph of $f(x) = 2^x$ is as indicated in Figure 1. We also can show that for x irrational, b^x can be approximated as closely as we like by using rational number approximations for x. Because $\sqrt{2} = 1.414213\ldots$, for example, the sequence

$$2^{1.4}, 2^{1.41}, 2^{1.414}, \ldots$$

approximates $2^{\sqrt{2}}$, and as we use more decimal places, the approximation improves.

› DEFINITION 1 Exponential Function

The equation

$$f(x) = b^x \qquad b > 0, b \neq 1$$

defines an **exponential function** for each different constant b, called the **base.** The independent variable x may assume any real value.

Thus, the **domain** of an exponential function is the set of all real numbers, and it can be shown that the **range** of an exponential function is the set of all positive real numbers. We require the base b to be positive to avoid imaginary numbers such as $(-2)^{1/2}$.

› Graphs of Exponential Functions

››› EXPLORE-DISCUSS 1

Compare the graphs of $f(x) = 3^x$ and $g(x) = 2^x$ by plotting both functions on the same coordinate system. Find all points of intersection of the graphs. For which values of x is the graph of f above the graph of g? Below the graph of g? Are the graphs of f and g close together as $x \to \infty$? As $x \to -\infty$? Discuss.

It is useful to compare the graphs of $y = 2^x$ and $y = (\frac{1}{2})^x = 2^{-x}$ by plotting both on the same coordinate system, as shown in Figure 2(a). The graph of

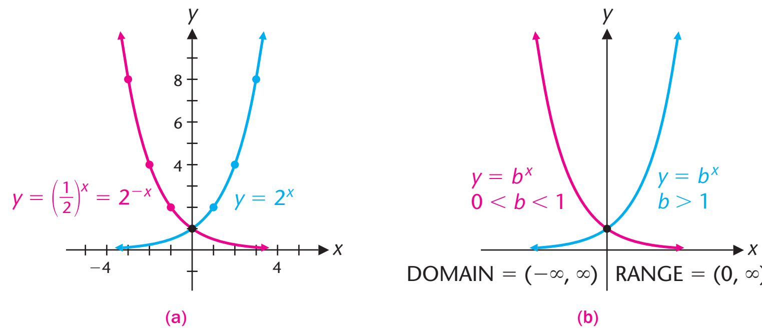
$$f(x) = b^x \qquad b > 1 \qquad \text{Fig. 2(b)}$$

looks very much like the graph of the particular case $y = 2^x$, and the graph of

$$f(x) = b^x \qquad 0 < b < 1 \qquad \text{Fig. 2(b)}$$

looks very much like the graph of $y = (\frac{1}{2})^x$. Note in both cases that the x axis is a *horizontal asymptote* for the graph.

Figure 2 Basic exponential graphs.



The graphs in Figure 2 suggest that the graphs of exponential functions have the properties listed in Theorem 1, which we state without proof.

THEOREM 1 Properties of Graphs of Exponential Functions

Let $f(x) = b^x$ be an exponential function, $b > 0$, $b \neq 1$. Then the graph of $f(x)$:

1. Is continuous for all real numbers
2. Has no sharp corners
3. Passes through the point (0, 1)
4. Lies above the x axis, which is a horizontal asymptote
5. Increases as x increases if $b > 1$; decreases as x increases if $0 < b < 1$
6. Intersects any horizontal line at most once (that is, f is one-to-one)

Property 4 of Theorem 1 implies that the graph of an exponential function cannot be the graph of a polynomial function. Properties 4 and 5 together imply that the graph of an exponential function cannot be the graph of a rational function. Property 6 implies that exponential functions have inverses; those inverses, called *logarithmic functions,* are discussed in Section 5-3.

Transformations of exponential functions are used to model population growth and radioactive decay (these applications and others are discussed in Section 5-2). It is important to understand how the graphs of those transformations are related to the graphs of the exponential functions. We explain such a relationship in Example 1 using the terminology of graph transformations from Section 3-5.

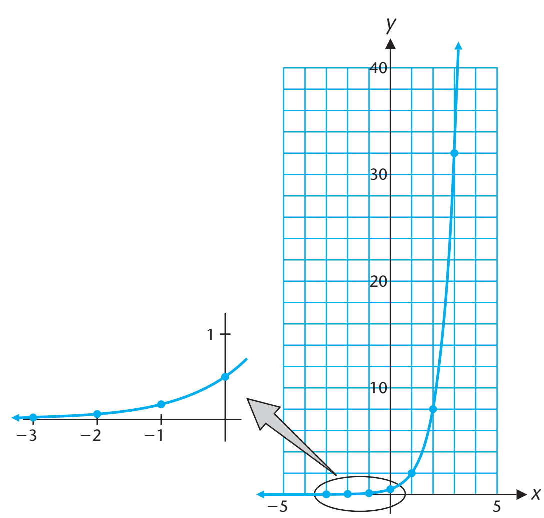
EXAMPLE 1

Transformations of Exponential Functions

Let $g(x) = \frac{1}{2}(4^x)$. Use transformations to explain how the graph of g is related to the graph of the exponential function $f(x) = 4^x$. Find the intercepts and asymptotes, and sketch the graph of g.

SOLUTION

The graph of g is a vertical shrink of the graph of f by a factor of $\frac{1}{2}$. Therefore $g(x) > 0$ for all real numbers and $g(x) \to 0$ as $x \to -\infty$. The x axis is a horizontal asymptote, $\frac{1}{2}$ is the y intercept, and there is no x intercept. We plot the intercept and some additional points and sketch the graph of g (Fig. 3).



› **Figure 3**

MATCHED PROBLEM 1

Let $g(x) = \frac{1}{2}(4^{-x})$. Use transformations to explain how the graph of g is related to the graph of the exponential function $f(x) = 4^x$. Find the intercepts and asymptotes, and sketch the graph of g.

Additional Exponential Properties

Exponential functions whose domains include irrational numbers obey the familiar laws of exponents for rational exponents. We summarize these exponent laws here and add two other important and useful properties.

EXPONENTIAL FUNCTION PROPERTIES

For a and b positive, $a \neq 1$, $b \neq 1$, and x and y real:

1. Exponent laws:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \frac{2^{5x}}{2^{7x}} \boxed{= 2^{5x-7x}}^{*} = 2^{-2x}$$

2. $a^x = a^y$ if and only if $x = y$. If $6^{4x} = 6^{2x+4}$, then $4x = 2x + 4$, and $x = 2$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$. If $a^4 = 3^4$, then $a = 3$.

Property 2 is another way to express the fact the exponential function $f(x) = a^x$ is one-to-one (see property 6 of Theorem 1). Because all exponential functions pass through the point (0, 1) (see property 3 of Theorem 1), property 3 indicates that the graphs of exponential functions with different bases do not intersect at any other points.

EXAMPLE 2 Using Exponential Function Properties

Solve $4^{x-3} = 8$ for x.

SOLUTION

Express both sides in terms of the same base, and use property 2 to equate exponents.

$$\begin{aligned} 4^{x-3} &= 8 && \text{Express 4 and 8 as powers of 2.} \\ (2^2)^{x-3} &= 2^3 && (a^x)^y = a^{xy} \\ 2^{2x-6} &= 2^3 && \text{Property 2} \\ 2x - 6 &= 3 && \text{Add 6 to both sides.} \\ 2x &= 9 && \text{Divide both sides by 2.} \\ x &= \tfrac{9}{2} \end{aligned}$$

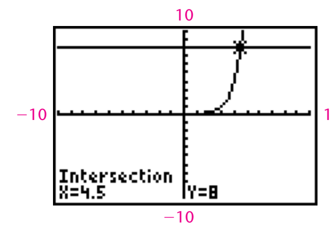
CHECK

$$4^{(9/2)-3} = 4^{3/2} = (\sqrt{4})^3 = 2^3 \stackrel{\checkmark}{=} 8$$

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

Technology Connections

As an alternative to the algebraic method of Example 2, you can use a graphing calculator to solve the equation $4^{x-3} = 8$. Graph $y_1 = 4^{x-3}$ and $y_2 = 8$, then use the intersect command to obtain $x = 4.5$ (Fig. 4).



› Figure 4

MATCHED PROBLEM 2

Solve $27^{x+1} = 9$ for x.

› Base e Exponential Function

Surprisingly, among the exponential functions it is not the function $g(x) = 2^x$ with base 2 or the function $h(x) = 10^x$ with base 10 that is used most frequently in mathematics. Instead, it is the function $f(x) = e^x$ with base e, where e is the limit of the expression

$$\left(1 + \frac{1}{x}\right)^x \tag{1}$$

as x gets larger and larger.

Table 1

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.593 74 . . .
100	2.704 81 . . .
1,000	2.716 92 . . .
10,000	2.718 14 . . .
100,000	2.718 27 . . .
1,000,000	2.718 28 . . .

››› EXPLORE-DISCUSS 2

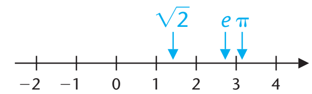
(A) Calculate the values of $[1 + (1/x)]^x$ for $x = 1, 2, 3, 4$, and 5. Are the values increasing or decreasing as x gets larger?

(B) Graph $y = [1 + (1/x)]^x$ and discuss the behavior of the graph as x increases without bound.

By calculating the value of expression (1) for larger and larger values of x (Table 1), it appears that $[1 + (1/x)]^x$ approaches a number close to 2.7183. In a calculus course we can show that as x increases without bound, the value of $[1 + (1/x)]^x$

approaches an irrational number that we call e. Just as irrational numbers such as π and $\sqrt{2}$ have unending, nonrepeating decimal representations, e also has an unending, nonrepeating decimal representation (see Section R-1). To 12 decimal places,

$$e = 2.718\ 281\ 828\ 459$$



Exactly who discovered e is still being debated. It is named after the great Swiss mathematician Leonhard Euler (1707–1783), who computed e to 23 decimal places using $[1 + (1/x)]^x$.

The constant e turns out to be an ideal base for an exponential function because in calculus and higher mathematics many operations take on their simplest form using this base. This is why you will see e used extensively in expressions and formulas that model real-world phenomena.

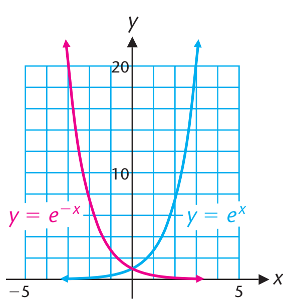


Figure 5
Exponential functions with base e.

DEFINITION 2 Exponential Function with Base e

For x a real number, the equation

$$f(x) = e^x$$

defines the **exponential function with base e.**

The exponential function with base e is used so frequently that it is often referred to as *the* exponential function. The graphs of $y = e^x$ and $y = e^{-x}$ are shown in Figure 5.

EXPLORE-DISCUSS 3

A graphing calculator was used to graph the functions $f(x) = 3^x$, $g(x) = 2^x$, and $h(x) = e^x$ in Figure 6. Where do the graphs intersect? Which graph lies between the others? Which graph is above the others when $x > 0$? When $x < 0$? Discuss the behavior of the three functions as $x \to \infty$ and as $x \to -\infty$.

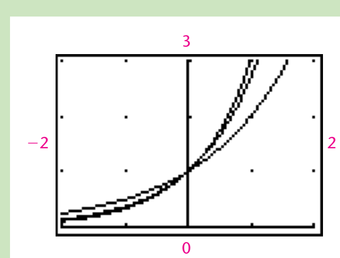


Figure 6

EXAMPLE 3 Analyzing a Graph

Let $g(x) = 4 - e^{x/2}$. Use transformations to explain how the graph of g is related to the graph of $f_1(x) = e^x$. Determine whether g is increasing or decreasing, find any asymptotes, and sketch the graph of g.

SOLUTION

The graph of g can be obtained from the graph of f_1 by a sequence of three transformations:

$$f_1(x) = e^x \underset{\text{Horizontal stretch}}{\longrightarrow} f_2(x) = e^{x/2} \underset{\text{Reflection in } x \text{ axis}}{\longrightarrow} f_3(x) = -e^{x/2} \underset{\text{Vertical translation}}{\longrightarrow} g(x) = 4 - e^{x/2}$$

[See Fig. 7(a) for the graphs of f_1, f_2, and f_3, and Fig. 7(b) for the graph of g.] The function g is decreasing for all x. Because $e^{x/2} \to 0$ as $x \to -\infty$, it follows that $g(x) = 4 - e^{x/2} \to 4$ as $x \to -\infty$. Therefore, the line $y = 4$ is a horizontal asymptote [indicated by the dashed line in Fig. 7(b)]; there are no vertical asymptotes. [To check that the graph of g (as obtained by graph transformations) is correct, plot a few points.]

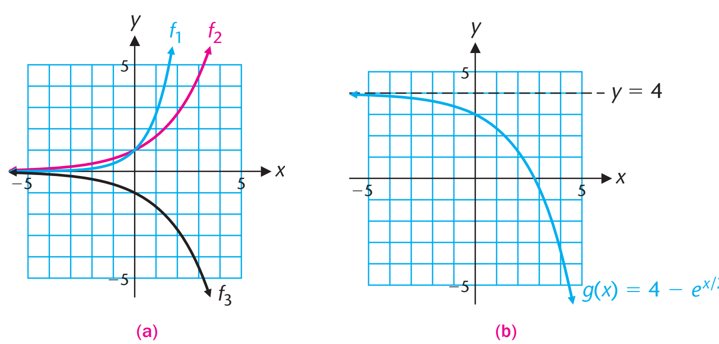


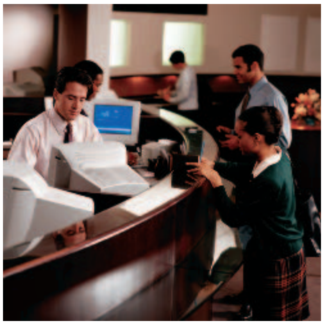
Figure 7

MATCHED PROBLEM 3

Let $g(x) = 2e^{x/2} - 5$. Use transformations to explain how the graph of g is related to the graph of $f_1(x) = e^x$. Describe the increasing/decreasing behavior, find any asymptotes, and sketch the graph of g.

Compound Interest

The fee paid to use another's money is called **interest.** It is usually computed as a percentage, called the **interest rate,** of the principal over a given time. If, at the end of a payment period, the interest due is reinvested at the same rate, then the interest earned as well as the principal will earn interest during the next payment period. Interest paid on interest reinvested is called **compound interest.**



Suppose you deposit \$1,000 in a savings and loan that pays 8% compounded semiannually. How much will the savings and loan owe you at the end of 2 years? Compounded semiannually means that interest is paid to your account at the end of each 6-month period, and the interest will in turn earn interest. The **interest rate per period** is the annual rate, 8% = 0.08, divided by the number of compounding periods per year, 2. If we let A_1, A_2, A_3, and A_4 represent the new amounts due at the end of the first, second, third, and fourth periods, respectively, then

$$\begin{aligned} A_1 &= \$1{,}000 + \$1{,}000\left(\frac{0.08}{2}\right) && \text{Factor out 1,000.} \\ &= \$1{,}000(1 + 0.04) && P\left(1 + \frac{r}{n}\right) \end{aligned}$$

$$\begin{aligned} A_2 &= A_1(1 + 0.04) && \text{Substitute for } A_1. \\ &= [\$1{,}000(1 + 0.04)](1 + 0.04) && \text{Associative law} \\ &= \$1{,}000(1 + 0.04)^2 && P\left(1 + \frac{r}{n}\right)^2 \end{aligned}$$

$$\begin{aligned} A_3 &= A_2(1 + 0.04) && \text{Substitute for } A_2. \\ &= [\$1{,}000(1 + 0.04)^2](1 + 0.04) && \text{Associative law} \\ &= \$1{,}000(1 + 0.04)^3 && P\left(1 + \frac{r}{n}\right)^3 \end{aligned}$$

$$\begin{aligned} A_4 &= A_3(1 + 0.04) && \text{Substitute for } A_3. \\ &= [\$1{,}000(1 + 0.04)^3](1 + 0.04) && \text{Associative law} \\ &= \$1{,}000(1 + 0.04)^4 && P\left(1 + \frac{r}{n}\right)^4 \end{aligned}$$

What do you think the savings and loan will owe you at the end of 6 years? If you guessed

$$A = \$1{,}000(1 + 0.04)^{12}$$

you have observed a pattern that is generalized in the following compound interest formula:

COMPOUND INTEREST

If a **principal** P is invested at an annual **rate** r compounded m times a year, then the **amount** A in the account at the end of n compounding periods is given by

$$A = P\left(1 + \frac{r}{m}\right)^n$$

The annual rate r is expressed in decimal form.

EXAMPLE 4 Compound Interest

If you deposit \$5,000 in an account paying 9% compounded daily,* how much will you have in the account in 5 years? Compute the answer to the nearest cent.

SOLUTION

We use the compound interest formula with $P = 5{,}000$, $r = 0.09$, $m = 365$, and $n = 5(365) = 1{,}825$:

$$A = P\left(1 + \frac{r}{m}\right)^n \qquad \text{Let } P = 5{,}000,\ r = 0.09,\ m = 365,\ n = 5(365)$$

$$= 5{,}000\left(1 + \frac{0.09}{365}\right)^{1825} \qquad \text{Calculate to nearest cent.}$$

$$= \$7{,}841.13$$

MATCHED PROBLEM 4

If \$1,000 is invested in an account paying 10% compounded monthly, how much will be in the account at the end of 10 years? Compute the answer to the nearest cent.

EXAMPLE 5 Comparing Investments

If \$1,000 is deposited into an account earning 10% compounded monthly and, at the same time, \$2,000 is deposited into an account earning 4% compounded monthly, will the first account ever be worth more than the second? If so, when?

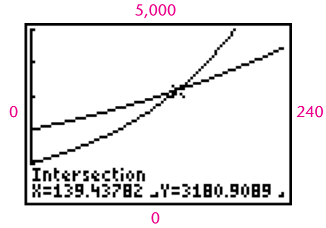
SOLUTION

Let y_1 and y_2 represent the amounts in the first and second accounts, respectively, then

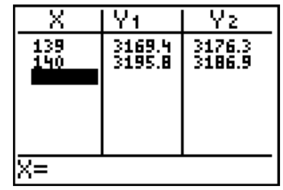
$$y_1 = 1{,}000(1 + 0.10/12)^x$$
$$y_2 = 2{,}000(1 + 0.04/12)^x$$

where x is the number of compounding periods (months). Examining the graphs of y_1 and y_2 [Fig. 8(a)], we see that the graphs intersect at $x \approx 139.438$ months. Because compound interest is paid at the end of each compounding period, we compare the amount in the accounts after 139 months and after 140 months [Fig. 8(b)]. Thus, the first account is worth more than the second for $x \geq 140$ months or 11 years and 8 months.

*In all problems involving interest that is compounded daily, we assume a 365-day year.



(a)



(b)

Figure 8

MATCHED PROBLEM 5

If \$4,000 is deposited into an account earning 10% compounded quarterly and, at the same time, \$5,000 is deposited into an account earning 6% compounded quarterly, when will the first account be worth more than the second?

Continuous Compound Interest

If \$100 is deposited in an account that earns compound interest at an annual rate of 8% for 2 years, how will the amount A change if the number of compounding periods is increased? If m is the number of compounding periods per year, then

$$A = 100\left(1 + \frac{0.08}{m}\right)^{2m}$$

The amount A is computed for several values of m in Table 2. Notice that the largest gain appears in going from annually to semiannually. Then, the gains slow down as m increases. In fact, it appears that A might be tending to something close to \$117.35 as m gets larger and larger.

Table 2 Effect of Compounding Frequency

Compounding Frequency	m	$A = 100\left(1 + \frac{0.08}{m}\right)^{2m}$
Annually	1	\$116.6400
Semiannually	2	116.9859
Quarterly	4	117.1659
Weekly	52	117.3367
Daily	365	117.3490
Hourly	8,760	117.3501

We now return to the general problem to see if we can determine what happens to $A = P[1 + (r/m)]^{mt}$ as m increases without bound. A little algebraic manipulation of the compound interest formula will lead to an answer and a significant result in the mathematics of finance:

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \qquad \text{Change algebraically.}$$

$$= P\left(1 + \frac{1}{m/r}\right)^{(m/r)rt} \qquad \text{Let } x = m/r.$$

$$= P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

The expression within the square brackets should look familiar. Recall from the first part of this section that

$$\left(1 + \frac{1}{x}\right)^{x} \to e \qquad \text{as} \qquad x \to \infty$$

Because r is fixed, $x = m/r \to \infty$ as $m \to \infty$. Thus,

$$P\left(1 + \frac{r}{m}\right)^{mt} \to Pe^{rt} \qquad \text{as} \qquad m \to \infty$$

and we have arrived at the **continuous compound interest formula,** a very important and widely used formula in business, banking, and economics.

› CONTINUOUS COMPOUND INTEREST FORMULA

If a principal P is invested at an annual rate r compounded continuously, then the amount A in the account at the end of t years is given by

$$A = Pe^{rt}$$

The annual rate r is expressed as a decimal.

EXAMPLE 6 Continuous Compound Interest

If \$100 is invested at an annual rate of 8% compounded continuously, what amount, to the nearest cent, will be in the account after 2 years?

SOLUTION

Use the continuous compound interest formula to find A when $P = \$100$, $r = 0.08$, and $t = 2$:

$$A = Pe^{rt} \qquad \text{8\% is equivalent to } r = 0.08.$$

$$= \$100e^{(0.08)(2)} \qquad \text{Calculate to nearest cent.}$$

$$= \$117.35$$

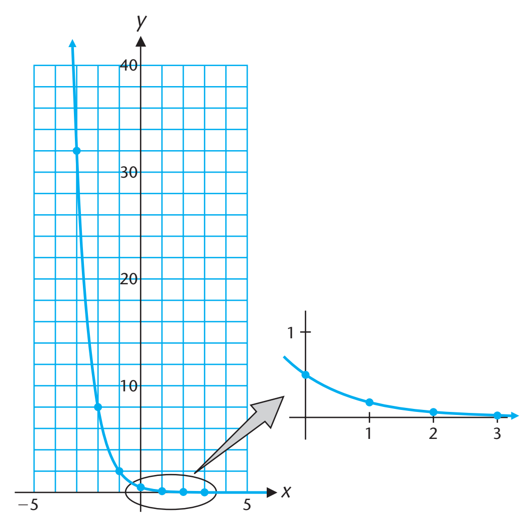
Compare this result with the values calculated in Table 2.

MATCHED PROBLEM 6

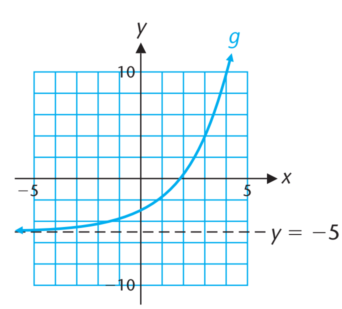
What amount will an account have after 5 years if $100 is invested at an annual rate of 12% compounded annually? Quarterly? Continuously? Compute answers to the nearest cent.

ANSWERS TO MATCHED PROBLEMS

1. The graph of g is the same as the graph of f reflected in the y axis and vertically shrunk by a factor of $\frac{1}{2}$.
   x intercepts: none
   y intercept: $\frac{1}{2}$
   horizontal asymptote: $y = 0$ (x axis)
   vertical asymptotes: none



2. $x = -\frac{1}{3}$
3. The graph of g is the same as the graph of f_1 stretched horizontally by a factor of 2, stretched vertically by a factor of 2, and shifted 5 units down; g is increasing.
   horizontal asymptote: $y = -5$
   vertical asymptote: none

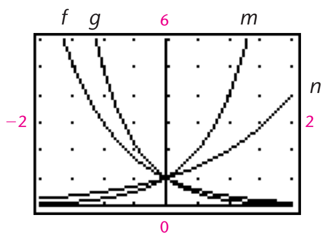


4. $2,707.04
5. After 23 quarters
6. Annually: $176.23; quarterly: $180.61; continuously: $182.21

5-1 Exercises

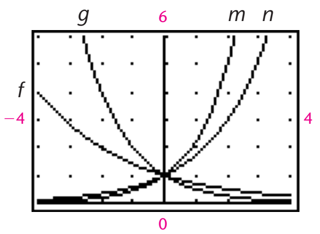
1. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = (0.2)^x$ (B) $y = 2^x$
(C) $y = (\frac{1}{3})^x$ (D) $y = 4^x$



2. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = e^{-1.2x}$ (B) $y = e^{0.7x}$
(C) $y = e^{-0.4x}$ (D) $y = e^{1.3x}$



In Problems 3–10, compute answers to four significant digits.

3. $5^{\sqrt{3}}$ **4.** $3^{-\sqrt{2}}$

5. $e^2 + e^{-2}$ **6.** $e - e^{-1}$

7. $\sqrt{e}$ **8.** $e^{\sqrt{2}}$

9. $\dfrac{2^\pi + 2^{-\pi}}{2}$ **10.** $\dfrac{3^\pi - 3^{-\pi}}{2}$

In Problems 11–22, simplify.

11. $2^e 5^e$ **12.** $e^2 e^5$

13. $(e^{\sqrt{2}})^{\sqrt{2}}$ **14.** $\dfrac{16^{\pi/4}}{2^\pi}$

15. $10^{3x-1}10^{4-x}$ **16.** $(4^{3x})^{2y}$

17. $\dfrac{3^x}{3^{1-x}}$ **18.** $\dfrac{5^{x-3}}{5^{x-4}}$

19. $\left(\dfrac{4^x}{5^y}\right)^{3z}$ **20.** $(2^x 3^y)^z$

21. $\dfrac{e^{5x}}{e^{2x+1}}$ **22.** $\dfrac{e^{4-3x}}{e^{2-5x}}$

In Problems 23–32, find the equations of any horizontal asymptotes without graphing.

23. $y = 4^x$ **24.** $y = 5^{-x}$

25. $y = 2 + e^{-x}$ **26.** $y = e^x - 4$

27. $f(t) = 2e^{3t} - 5$ **28.** $g(t) = 6 - 7e^{2t}$

29. $M(x) = 1 - e^{-x^2}$ **30.** $N(x) = e^{x^2} - 3$

31. $R(t) = 3e^{t^2} - 8$ **32.** $S(t) = 9 - 5e^{-t^2}$

In Problems 33–42, use transformations to explain how the graph of g is related to the graph of $f(x) = e^x$. Determine whether g is increasing or decreasing, find the asymptotes, and sketch the graph of g.

33. $g(x) = 3e^x$ **34.** $g(x) = 2e^{-x}$

35. $g(x) = \frac{1}{3}e^{-x}$ **36.** $g(x) = \frac{1}{5}e^x$

37. $g(x) = 2 + e^x$ **38.** $g(x) = -4 + e^x$

39. $g(x) = -2e^x$ **40.** $g(x) = -3e^x$

41. $g(x) = e^{x+2}$ **42.** $g(x) = e^{x-1}$

In Problems 43–66, solve for x.

43. $5^{3x} = 5^{4x-2}$ **44.** $10^{2-3x} = 10^{5x-6}$

45. $7^{x^2} = 7^{2x+3}$ **46.** $4^{5x-x^2} = 4^{-6}$

47. $(\frac{1}{2})^{x+4} = (\frac{1}{2})^{3x-5}$ **48.** $(\frac{1}{3})^{2x-1} = (\frac{1}{3})^{3-x}$

49. $(\frac{4}{5})^{6x+1} = \frac{5}{4}$ **50.** $(\frac{7}{3})^{2-x} = \frac{3}{7}$

51. $(1 - x)^5 = (2x - 1)^5$ **52.** $5^3 = (x + 2)^3$

53. $2xe^{-x} = 0$ **54.** $(x - 3)e^x = 0$

55. $x^2e^x - 5xe^x = 0$ **56.** $3xe^{-x} + x^2e^{-x} = 0$

57. $9^{x^2} = 3^{3x-1}$ **58.** $4^{x^2} = 2^{x+3}$

59. $25^{x+3} = 125^x$ **60.** $4^{5x+1} = 16^{2x-1}$

61. $4^{2x+7} = 8^{x+2}$

62. $100^{2x+3} = 1{,}000^{x+5}$

63. $100^{x^2} = 1{,}000^{10}$

64. $27^{4x} = 81^{100}$

65. $(\frac{1}{9})^{5x+1} = 27$

66. $(\frac{1}{8})^{4x} = 16^{x+3}$

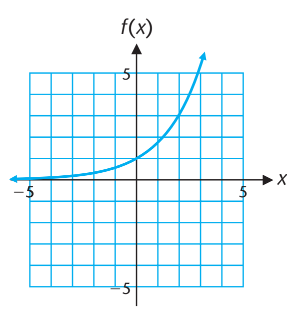
67. Find all real numbers a such that $a^2 = a^{-2}$. Explain why this does not violate the second exponential function property in the box on page 458.

68. Find real numbers a and b such that $a \neq b$ but $a^4 = b^4$. Explain why this does not violate the third exponential function property in the box on page 458.

69. Examine the graph of $y = 1^x$ on a graphing utility and explain why 1 cannot be the base for an exponential function.

70. Examine the graph of $y = 0^x$ on a graphing utility and explain why 0 cannot be the base for an exponential function. [*Hint:* Turn the axes off before graphing.]

Graph each function in Problems 71–78 using the graph of f shown in the figure.



71. $y = f(x) - 2$

72. $y = f(x) + 1$

73. $y = f(x - 2)$

74. $y = f(x + 1)$

75. $y = 2f(x) - 4$

76. $y = 3 - 5f(x)$

77. $y = 2 - 3f(x - 4)$

78. $y = 2f(x + 1) - 1$

In Problems 79–88, use transformations to explain how the graph of g is related to the graph of the given exponential function f. Determine whether g is increasing or decreasing, find any asymptotes, and sketch the graph of g.

79. $g(x) = -(\frac{1}{2})^x$; $f(x) = (\frac{1}{2})^x$

80. $g(x) = -(\frac{1}{3})^{-x}$; $f(x) = (\frac{1}{3})^x$

81. $g(x) = (\frac{1}{4})^{x/2} + 3$; $f(x) = (\frac{1}{4})^x$

82. $g(x) = 5 - (\frac{2}{3})^{3x}$; $f(x) = (\frac{2}{3})^x$

83. $g(x) = 500(1.04)^x$; $f(x) = 1.04^x$

84. $g(x) = 1{,}000(1.03)^x$; $f(x) = 1.03^x$

85. $g(x) = 1 + 2e^{x-3}$; $f(x) = e^x$

86. $g(x) = 4e^{x+1} - 7$; $f(x) = e^x$

87. $g(x) = 3 - 4e^{2-x}$; $f(x) = e^x$

88. $g(x) = -2 - 5e^{4-x}$; $f(x) = e^x$

In Problems 89–92, simplify.

89. $\dfrac{-2x^3e^{-2x} - 3x^2e^{-2x}}{x^6}$

90. $\dfrac{5x^4e^{5x} - 4x^3e^{5x}}{x^8}$

91. $(e^x + e^{-x})^2 + (e^x - e^{-x})^2$

92. $e^x(e^{-x} + 1) - e^{-x}(e^x + 1)$

In Problems 93–104, use a graphing calculator to find local extrema, y intercepts, and x intercepts. Investigate the behavior as $x \to \infty$ and as $x -\infty$ and identify and horizontal asymptotes. Round any approximate values to two decimal places.

93. $f(x) = 2 + e^{x-2}$

94. $g(x) = -3 + e^{1+x}$

95. $m(x) = e^{|x|}$

96. $n(x) = e^{-|x|}$

97. $s(x) = e^{-x^2}$

98. $r(x) = e^{x^2}$

99. $F(x) = \dfrac{200}{1 + 3e^{-x}}$

100. $G(x) = \dfrac{100}{1 + e^{-x}}$

101. $m(x) = 2x(3^{-x}) + 2$

102. $h(x) = 3x(2^{-x}) - 1$

103. $f(x) = \dfrac{2^x + 2^{-x}}{2}$

104. $g(x) = \dfrac{3^x + 3^{-x}}{2}$

105. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches 0.

106. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches ∞.

It is common practice in many applications of mathematics to approximate nonpolynomial functions with appropriately selected polynomials. For example, the polynomials in Problems 107–110, called ***Taylor polynomials,*** *can be used to approximate the exponential function $f(x) = e^x$. To illustrate this approximation graphically, in each problem graph $f(x) = e^x$ and the indicated polynomial in the same viewing window, $-4 \le x \le 4$ and $-5 \le y \le 50$.*

107. $P_1(x) = 1 + x + \frac{1}{2}x^2$

108. $P_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

109. $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

110. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

111. Investigate the behavior of the functions $f_1(x) = x/e^x$, $f_2(x) = x^2/e^x$, and $f_3(x) = x^3/e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $f_n(x) = x^n/e^x$, where n is any positive integer.

112. Investigate the behavior of the functions $g_1(x) = xe^x$, $g_2(x) = x^2e^x$, and $g_3(x) = x^3e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $g_n(x) = x^n e^x$, where n is any positive integer.

113. Explain why the graph of an exponential function cannot be the graph of a polynomial function.

114. Explain why the graph of an exponential function cannot be the graph of a rational function.

APPLICATIONS*

★**115.** FINANCE A couple just had a new child. How much should they invest now at 8.25% compounded daily to have $40,000 for the child's education 17 years from now? Compute the answer to the nearest dollar.

★**116.** FINANCE A person wishes to have $15,000 cash for a new car 5 years from now. How much should be placed in an account now if the account pays 9.75% compounded weekly? Compute the answer to the nearest dollar.

117. MONEY GROWTH If you invest $5,250 in an account paying 11.38% compounded continuously, how much money will be in the account at the end of
(A) 6.25 years? (B) 17 years?

118. MONEY GROWTH If you invest $7,500 in an account paying 8.35% compounded continuously, how much money will be in the account at the end of
(A) 5.5 years? (B) 12 years?

★**119.** FINANCE If $3,000 is deposited into an account earning 8% compounded daily and, at the same time, $5,000 is deposited into an account earning 5% compounded daily, will the first account be worth more than the second? If so, when?

★**120.** FINANCE If $4,000 is deposited into an account earning 9% compounded weekly and, at the same time, $6,000 is deposited into an account earning 7% compounded weekly, will the first account be worth more than the second? If so, when?

★**121.** FINANCE Will an investment of $10,000 at 8.9% compounded daily ever be worth more at the end of any quarter than an investment of $10,000 at 9% compounded quarterly? Explain.

★**122.** FINANCE A sum of $5,000 is invested at 13% compounded semiannually. Suppose that a second investment of $5,000 is made at interest rate r compounded daily. Both investments are held for 1 year. For which values of r, to the nearest tenth of a percent, is the second investment better than the first? Discuss.

★**123.** PRESENT VALUE A promissory note will pay $30,000 at maturity 10 years from now. How much should you pay for the note now if the note gains value at a rate of 9% compounded continuously?

★**124.** PRESENT VALUE A promissory note will pay $50,000 at maturity $5\frac{1}{2}$ years from now. How much should you pay for the note now if the note gains value at a rate of 10% compounded continuously?

125. MONEY GROWTH *Barron's,* a national business and financial weekly, published the following "Top Savings Deposit Yields" for $2\frac{1}{2}$-year certificate of deposit accounts:

Gill Savings	8.30% (CC)
Richardson Savings and Loan	8.40% (CQ)
USA Savings	8.25% (CD)

where CC represents compounded continuously, CQ compounded quarterly, and CD compounded daily. Compute the value of $1,000 invested in each account at the end of $2\frac{1}{2}$ years.

126. MONEY GROWTH Refer to Problem 125. In another issue of *Barron's,* 1-year certificate of deposit accounts included:

Alamo Savings	8.25% (CQ)
Lamar Savings	8.05% (CC)

Compute the value of $10,000 invested in each account at the end of 1 year.

127. FINANCE Suppose $4,000 is invested at 11% compounded weekly. How much money will be in the account in
(A) $\frac{1}{2}$ year? (B) 10 years?

Compute answers to the nearest cent.

128. FINANCE Suppose $2,500 is invested at 7% compounded quarterly. How much money will be in the account in
(A) $\frac{3}{4}$ year? (B) 15 years?

Compute answers to the nearest cent.

*Round monetary amounts to the nearest cent unless specified otherwise. In all problems involving interest that is compounded daily, assume a 365-day year.

5-2 Exponential Models

- Mathematical Modeling
- Data Analysis and Regression
- A Comparison of Exponential Growth Phenomena

In Section 5-2 we use exponential functions to model a wide variety of real-world phenomena, including growth of populations of people, animals, and bacteria; radioactive decay; spread of epidemics; propagation of rumors; light intensity; atmospheric pressure; and electric circuits. The regression techniques introduced in Chapters 2 and 3 to construct linear and quadratic models are extended to construct exponential models.

Mathematical Modeling

Populations tend to grow exponentially and at different rates. A convenient and easily understood measure of growth rate is the **doubling time**—that is, the time it takes for a population to double. Over short periods the **doubling time growth model** is often used to model population growth:

$$P = P_0 2^{t/d}$$

where P = Population at time t

P_0 = Population at time $t = 0$

d = Doubling time

Note that when $t = d$,

$$P = P_0 2^{d/d} = P_0 2$$

and the population is double the original, as it should be. We use this model to solve a population growth problem in Example 1.

EXAMPLE 1 Population Growth

Nicaragua has a population of approximately 6 million and it is estimated that the population will double in 36 years. If population growth continues at the same rate, what will be the population:

(A) 15 years from now? (B) 40 years from now?

SOLUTIONS

We use the doubling time growth model:

$$P = P_0 2^{t/d}$$

Substituting $P_0 = 6$ and $d = 36$, we obtain

$$P = 6(2^{t/36})$$ Figure 1

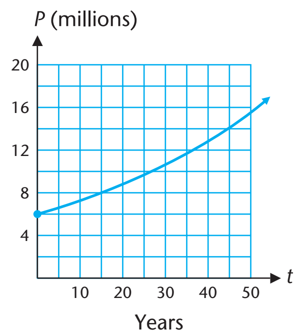


Figure 1
$P = 6(2^{t/36})$.

(A) Find P when $t = 15$ years:

$$P = 6(2^{15/36}) \approx 8 \text{ million}$$

(B) Find P when $t = 40$ years:

$$P = 6(2^{40/36}) \approx 13 \text{ million}$$

MATCHED PROBLEM 1

The bacterium *Escherichia coli* (*E. coli*) is found naturally in the intestines of many mammals. In a particular laboratory experiment, the doubling time for *E. coli* is found to be 25 minutes. If the experiment starts with a population of 1,000 *E. coli* and there is no change in the doubling time, how many bacteria will be present:

(A) In 10 minutes? (B) In 5 hours?

Write answers to three significant digits.

››› EXPLORE-DISCUSS 1

The doubling time growth model would *not* be expected to give accurate results over long periods. According to the doubling time growth model of Example 1, what was the population of Nicaragua 500 years ago when it was settled as a Spanish colony? What will the population of Nicaragua be 200 years from now? Explain why these results are unrealistic. Discuss factors that affect human populations that are not taken into account by the doubling time growth model.

As an alternative to the doubling time growth model, we can use the equation

$$y = ce^{kt}$$

where y = Population at time t
c = Population at time 0
k = Relative growth rate

The **relative growth rate** k has the following interpretation: Suppose that $y = ce^{kt}$ models the population growth of a country, where y is the number of persons and t is time in years. If the relative growth rate is $k = 0.03$, then at any time t, the population is growing at a rate of $0.03y$ persons (that is, 3% of the population) per year. Example 2 illustrates this approach.

EXAMPLE 2 Medicine—Bacteria Growth

Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as modeled by

$$N = N_0e^{1.386t}$$

where N is the number of bacteria present after t hours and N_0 is the number of bacteria present at $t = 0$. If we start with 1 bacterium, how many bacteria will be present in

(A) 5 hours? (B) 12 hours?

Compute the answers to three significant digits.

SOLUTIONS

(A) Use $N_0 = 1$ and $t = 5$:

$$\begin{aligned} N &= N_0e^{1.386t} && \text{Let } N_0 = 1 \text{ and } t = 5. \\ &= e^{1.386(5)} && \text{Calculate to three significant digits.} \\ &= 1{,}020 \end{aligned}$$

(B) Use $N_0 = 1$ and $t = 12$:

$$\begin{aligned} N &= N_0 e^{1.386t} && \text{Let } N_0 = 1 \text{ and } t = 12. \\ &= e^{1.386(12)} && \text{Calculate to three significant digits.} \\ &= 16{,}700{,}000 \end{aligned}$$

MATCHED PROBLEM 2

Repeat Example 2 if $N = N_0 e^{0.783t}$ and all other information remains the same.

Exponential functions can also be used to model radioactive decay, which is sometimes referred to as *negative growth.* Radioactive materials are used extensively in medical diagnosis and therapy, as power sources in satellites, and as power sources in many countries. If we start with an amount A_0 of a particular radioactive isotope, the amount declines exponentially in time. The rate of decay varies from isotope to isotope. A convenient and easily understood measure of the rate of decay is the **half-life** of the isotope—that is, the time it takes for half of a particular material to decay. We use the following **half-life decay model:**

$$\begin{aligned} A &= A_0\left(\tfrac{1}{2}\right)^{t/h} \\ &= A_0 2^{-t/h} \end{aligned}$$

where A = Amount at time t

A_0 = Amount at time $t = 0$

h = Half-life

Note that when $t = h$,

$$A = A_0 2^{-h/h} = A_0 2^{-1} = \frac{A_0}{2}$$

and the amount of isotope is half the original amount, as it should be.

EXAMPLE 3 Radioactive Decay

The radioactive isotope gallium 67 (^{67}Ga), used in the diagnosis of malignant tumors, has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after

(A) 24 hours? (B) 1 week?

Compute answers to three significant digits.

SOLUTIONS

We use the half-life decay model:

$$A = A_0\left(\tfrac{1}{2}\right)^{t/h} = A_0 2^{-t/h}$$

Using $A_0 = 100$ and $h = 46.5$, we obtain

$$A = 100(2^{-t/46.5}) \quad \text{Figure 2}$$

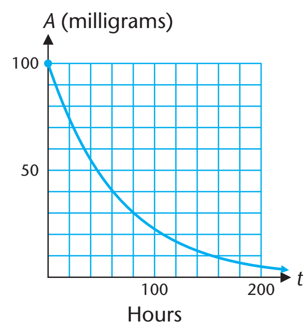


Figure 2
$A = 100(2^{-t/46.5})$.

(A) Find A when $t = 24$ hours:

$$\begin{aligned} A &= 100(2^{-24/46.5}) \quad \text{Calculate to three significant digits.} \\ &= 69.9 \text{ milligrams} \end{aligned}$$

(B) Find A when $t = 168$ hours
(1 week = 168 hours):

$$\begin{aligned} A &= 100(2^{-168/46.5}) \quad \text{Calculate to three significant digits.} \\ &= 8.17 \text{ milligrams} \end{aligned}$$

MATCHED PROBLEM 3

Radioactive gold 198 (^{198}Au), used in imaging the structure of the liver, has a half-life of 2.67 days. If we start with 50 milligrams of the isotope, how many milligrams will be left after:

(A) $\frac{1}{2}$ day? (B) 1 week?

Compute answers to three significant digits.

As an alternative to the half-life decay model, we can use the equation $y = ce^{-kt}$, where c and k are positive constants, to model radioactive decay. Example 4 illustrates this approach.

EXAMPLE 4 Carbon-14 Dating

Cosmic-ray bombardment of the atmosphere produces neutrons, which in turn react with nitrogen to produce radioactive carbon-14. Radioactive carbon-14 enters all living tissues through carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, carbon-14 is maintained in the living organism at a constant level. Once the organism dies, however, carbon-14 decays according to the equation

$$A = A_0 e^{-0.000124t}$$

where A is the amount of carbon-14 present after t years and A_0 is the amount present at time $t = 0$. If 1,000 milligrams of carbon-14 are present at the start, how many milligrams will be present in

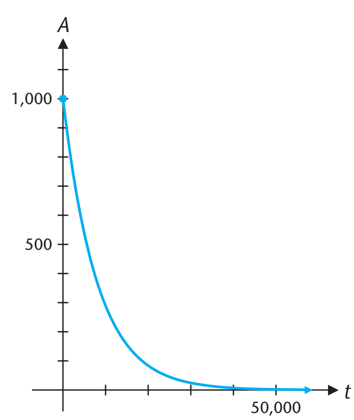
(A) 10,000 years? (B) 50,000 years?

Compute answers to three significant digits.

SOLUTIONS

Substituting $A_0 = 1{,}000$ in the decay equation, we have

$$A = 1{,}000e^{-0.000124t}$$ Figure 3



› Figure 3

(A) Solve for A when $t = 10{,}000$:

$$\begin{aligned} A &= 1{,}000e^{-0.000124(10{,}000)} \\ &= 289 \text{ milligrams} \end{aligned}$$ Calculate to three significant digits.

(B) Solve for A when $t = 50{,}000$:

$$\begin{aligned} A &= 1{,}000e^{-0.000124(50{,}000)} \\ &= 2.03 \text{ milligrams} \end{aligned}$$ Calculate to three significant digits.

More will be said about carbon-14 dating in Exercise 5-5, where we will be interested in solving for t after being given information about A and A_0.

MATCHED PROBLEM 4

Referring to Example 4, how many milligrams of carbon-14 would have to be present at the beginning to have 10 milligrams present after 20,000 years? Compute the answer to four significant digits.

We can model phenomena such as learning curves, for which growth has an upper bound, by the equation $y = c(1 - e^{-kt})$, where c and k are positive constants. Example 5 illustrates such limited growth.

EXAMPLE 5 Learning Curve

People assigned to assemble circuit boards for a computer manufacturing company undergo on-the-job training. From past experience, it was found that the learning curve for the average employee is given by

$$N = 40(1 - e^{-0.12t})$$

where N is the number of boards assembled per day after t days of training (Fig. 4).

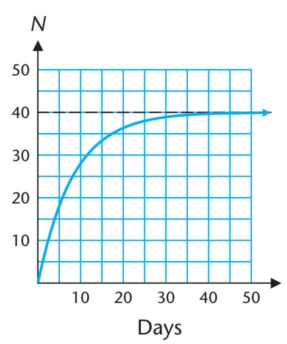


Figure 4
$N = 40(1 - e^{-0.12t})$.

(A) How many boards can an average employee produce after 3 days of training? After 5 days of training? Round answers to the nearest integer.

(B) Does N approach a limiting value as t increases without bound? Explain.

SOLUTION

(A) When $t = 3$,

$$N = 40(1 - e^{-0.12(3)}) = 12 \quad \text{Rounded to nearest integer}$$

so the average employee can produce 12 boards after 3 days of training. Similarly, when $t = 5$,

$$N = 40(1 - e^{-0.12(5)}) = 18 \quad \text{Rounded to nearest integer}$$

Because $e^{-0.12t}$ approaches 0 as t increases without bound,

$$N = 40(1 - e^{-0.12t}) \rightarrow 40(1 - 0) = 40$$

So the limiting value of N is 40 boards per day. (Note the horizontal asymptote with equation $N = 40$ that is indicated by the dashed line in Fig. 4.)

MATCHED PROBLEM 5

A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million potential viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

(A) How many viewers are aware of the product after 2 days? After 10 days? Express answers as integers, rounded to three significant digits.

(B) Does N approach a limiting value as t increases without bound? Explain.

We can model phenomena such as the spread of an epidemic or the propagation of a rumor by the *logistic equation*.

$$y = \frac{M}{(1 + ce^{-kt})}$$

where M, c, and k are positive constants. Logistic growth, illustrated in Example 6, approaches a limiting value as t increases without bound.

EXAMPLE 6 Logistic Growth in an Epidemic

A community of 1,000 individuals is assumed to be homogeneously mixed. One individual who has just returned from another community has influenza. Assume the community has not had influenza shots and all are susceptible. The spread of the disease in the community is predicted to be given by the logistic curve

$$N(t) = \frac{1{,}000}{1 + 999e^{-0.3t}}$$

where N is the number of people who have contracted influenza after t days (Fig. 5).

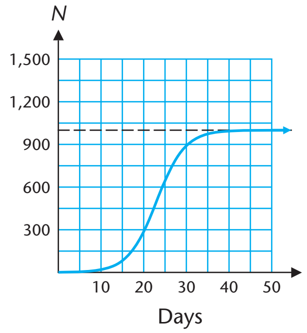


Figure 5

$N = \dfrac{1{,}000}{1 + 999e^{-0.3t}}$.

(A) How many people have contracted influenza after 10 days? After 20 days? Round answers to the nearest integer?

(B) Does N approach a limiting value as t increases without bound? Explain.

SOLUTIONS

(A) When $t = 10$,

$$N = \frac{1{,}000}{1 + 999e^{-0.3(10)}} = 20 \quad \text{Rounded to nearest integer}$$

so 20 people have contracted influenza after 10 days. Similarly, when $t = 20$,

$$N = \frac{1{,}000}{1 + 999e^{-0.3(20)}} = 288 \quad \text{Rounded to nearest integer}$$

so 288 people have contracted influenza after 20 days.

(B) Because $e^{-0.3t}$ approaches 0 as t increases without bound,

$$N = \frac{1{,}000}{1 + 999e^{-0.3t}} \rightarrow \frac{1{,}000}{1 + 999(0)} = 1{,}000$$

So the limiting value is 1,000 individuals (all in the community will eventually contract influenza). (Note the horizontal asymptote with equation $N = 1{,}000$ that is indicated by the dashed line in Fig. 5.)

MATCHED PROBLEM 6

A group of 400 parents, relatives, and friends are waiting anxiously at Kennedy Airport for a charter flight returning students after a year in Europe. It is stormy and the plane is late. A particular parent thought he had heard that the plane's radio had gone out and related this news to some friends, who in turn passed it on to others. The propagation of this rumor is predicted to be given by

$$N(t) = \frac{400}{1 + 399e^{-0.4t}}$$

where N is the number of people who have heard the rumor after t minutes.

(A) How many people have heard the rumor after 10 minutes? After 20 minutes? Round answers to the nearest integer.

(B) Does N approach a limiting value as t increases without bound? Explain.

Data Analysis and Regression

We use exponential regression to fit a function of the form $y = ab^x$ to a set of data points, and logistic regression to fit a function of the form

$$y = \frac{c}{1 + ae^{-bx}}$$

to a set of data points. The techniques are similar to those introduced in Chapters 2 and 3 for linear and quadratic functions.

EXAMPLE 7 Infectious Diseases

The U.S. Department of Health and Human Services published the data in Table 1.

Table 1 Reported Cases of Infectious Diseases

Year	Mumps	Rubella
1970	104,953	56,552
1980	8,576	3,904
1990	5,292	1,125
1995	906	128
2000	323	152

An exponential model for the data on mumps is given by

$$N = 91{,}400(0.835)^t$$

where N is the number of reported cases of mumps and t is time in years with $t = 0$ representing 1970.

(A) Use the model to predict the number of reported cases of mumps in 2010.

(B) Compare the actual number of cases of mumps reported in 1980 to the number given by the model.

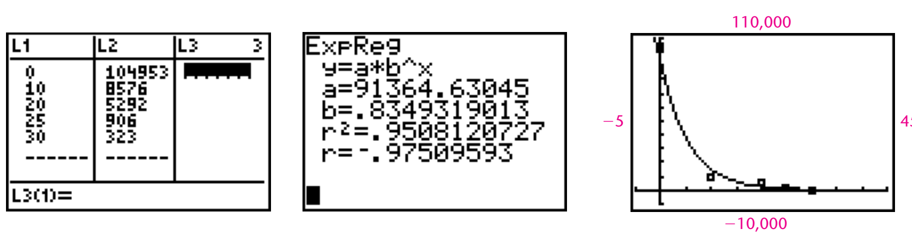
SOLUTIONS

(A) The year 2010 is represented by $t = 40$. Evaluating $N = 91{,}400(0.835)^t$ at $t = 40$ gives a prediction of 67 cases of mumps in 2010.

(B) The year 1980 is represented by $t = 10$. Evaluating $N = 91{,}400(0.835)^t$ at $t = 10$ gives 15,060 cases in 1980. The actual number of cases reported in 1980 was 8,576, nearly 6,500 less than the number given by the model.

Technology Connections

Figure 6 shows the details of constructing the exponential model of Example 7 on a graphing calculator.



(a) Entering the data (b) Finding the model (c) Graphing the data and the model

Figure 6

MATCHED PROBLEM 7

An exponential model for the data on rubella in Table 1 is given by

$$N = 44{,}500(0.815)^t$$

where N is the number of reported cases of rubella and t is time in years with $t = 0$ representing 1970.

(A) Use the model to predict the number of reported cases of rubella in 2010.

(B) Compare the actual number of cases of rubella reported in 1980 to the number given by the model.

EXAMPLE 8 AIDS Cases and Deaths

The U.S. Department of Health and Human Services published the data in Table 2.

Table 2 Acquired Immunodeficiency Syndrome (AIDS) Cases and Deaths in the United States

Year	Cases Diagnosed to Date	Known Deaths to Date
1985	23,185	12,648
1988	107,755	62,468
1991	261,259	159,294
1994	493,713	296,507
1997	672,970	406,179
2000	774,467	447,648
2003	929,985	524,060

A logistic model for the data on AIDS cases is given by

$$N = \frac{948,000}{1 + 17.8e^{-0.317t}}$$

where N is the number of AIDS cases diagnosed by year t with $t = 0$ representing 1985.

(A) Use the model to predict the number of AIDS cases diagnosed by 2010.

(B) Compare the actual number of AIDS cases diagnosed by 2003 to the number given by the model.

SOLUTIONS

(A) The year 2010 is represented by $t = 25$. Evaluating

$$N = \frac{948,000}{1 + 17.8e^{-0.317t}}$$

at $t = 25$ gives a prediction of approximately 942,000 cases of AIDS diagnosed by 2010.

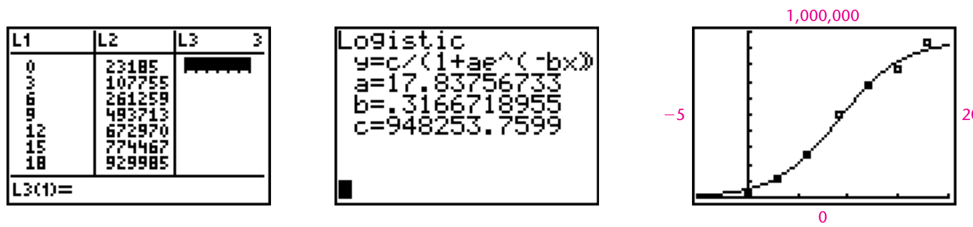
(B) The year 2003 is represented by $t = 18$. Evaluating

$$N = \frac{948,000}{1 + 17.8e^{-0.317t}}$$

at $t = 18$ gives 895,013 cases in 2003. The actual number of cases diagnosed by 2003 was 929,985, nearly 35,000 greater than the number given by the model.

Technology Connections

Figure 7 shows the details of constructing the logistic model of Example 7 on a graphing calculator.



(a) Entering the data (b) Finding the model (c) Graphing the data and the model

Figure 7

MATCHED PROBLEM 8

A logistic model for the data on deaths from AIDS in Table 2 is given by

$$N = \frac{520{,}000}{1 + 19.3e^{-0.353t}}$$

where N is the number of known deaths from AIDS by year t with $t = 0$ representing 1985.

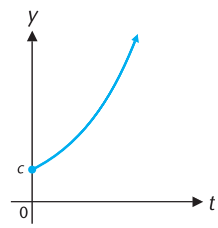
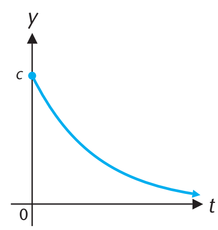
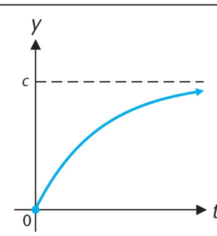
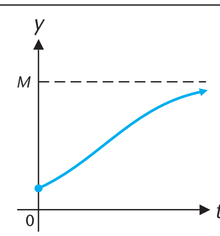
(A) Use the model to predict the number of known deaths from AIDS by 2010.

(B) Compare the actual number of known deaths from AIDS by 2003 to the number given by the model.

A Comparison of Exponential Growth Phenomena

The equations and graphs given in Table 3 compare the growth models discussed in Examples 1 through 8. Following each equation and graph is a short, incomplete list of areas in which the models are used. In the first case (unlimited growth), $y \to \infty$ as $t \to \infty$. In the other three cases (exponential decay, limited growth, and logistic growth), the graph approaches a horizontal asymptote as $t \to \infty$; these asymptotes ($y = 0$, $y = c$, and $y = M$, respectively) are easily deduced from the given equations. Table 3 only touches on a subject that you are likely to study in greater depth in the future.

Table 3 Exponential Growth and Decay

Description	Equation	Graph	Uses
Unlimited growth	$y = ce^{kt}$ $c, k > 0$		Short-term population growth (people, bacteria. etc.); growth of money at continuous compound interest
Exponential decay	$y = ce^{-kt}$ $c, k > 0$		Radioactive decay; light absorption in water, glass, and the like; atmospheric pressure; electric circuits
Limited growth	$y = c(1 - e^{-kt})$ $c, k > 0$		Learning skills; sales fads; company growth; electric circuits
Logistic growth	$y = \dfrac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth; epidemics; sales of new products; company growth

ANSWERS TO MATCHED PROBLEMS

1. (A) 1,320 bacteria (B) 4,100,100 = 4.10×10^6 bacteria
2. (A) 50 bacteria (B) 12,000 bacteria
3. (A) 43.9 milligrams (B) 8.12 milligrams **4.** 119.4 milligrams
5. (A) 143,000 viewers; 619,000 viewers
(B) N approaches an upper limit of 2 million, the number of potential viewers
6. (A) 48 individuals; 353 individuals
(B) N approaches an upper limit of 400, the number of people in the entire group.
7. (A) 12 cases
(B) The actual number of cases was 1,850 less than the number given by the model.
8. (A) 519,000 deaths
(B) The actual number of known deaths was approximately 21,000 greater than the number given by the model.

5-2 Exercises

APPLICATIONS

1. GAMING A person bets on red and black on a roulette wheel using a *Martingale strategy.* That is, a \$2 bet is placed on red, and the bet is doubled each time until a win occurs. The process is then repeated. If black occurs n times in a row, then $L = 2^n$ dollars is lost on the nth bet. Graph this function for $1 \le n \le 10$. Although the function is defined only for positive integers, points on this type of graph are usually joined with a smooth curve as a visual aid.

2. BACTERIAL GROWTH If bacteria in a certain culture double every $\frac{1}{2}$ hour, write an equation that gives the number of bacteria N in the culture after t hours, assuming the culture has 100 bacteria at the start. Graph the equation for $0 \le t \le 5$.

3. POPULATION GROWTH Because of its short life span and frequent breeding, the fruit fly *Drosophila* is used in some genetic studies. Raymond Pearl of Johns Hopkins University, for example, studied 300 successive generations of descendants of a single pair of *Drosophila* flies. In a laboratory situation with ample food supply and space, the doubling time for a particular population is 2.4 days. If we start with 5 male and 5 female flies, how many flies should we expect to have in
(A) 1 week? (B) 2 weeks?

4. POPULATION GROWTH If Kenya has a population of about 34,000,000 people and a doubling time of 27 years and if the growth continues at the same rate, find the population in
(A) 10 years (B) 30 years

Compute answers to 2 significant digits.

5. INSECTICIDES The use of the insecticide DDT is no longer allowed in many countries because of its long-term adverse effects. If a farmer uses 25 pounds of active DDT, assuming its half-life is 12 years, how much will still be active after
(A) 5 years? (B) 20 years?

Compute answers to two significant digits.

6. RADIOACTIVE TRACERS The radioactive isotope technetium-99m (^{99m}Tc) is used in imaging the brain. The isotope has a half-life of 6 hours. If 12 milligrams are used, how much will be present after
(A) 3 hours? (B) 24 hours?

Compute answers to three significant digits.

7. POPULATION GROWTH If the world population is about 6.5 billion people now and if the population grows continuously at a relative growth rate of 1.14%, what will the population be in 10 years? Compute the answer to two significant digits.

8. POPULATION GROWTH If the population in Mexico is around 106 million people now and if the population grows continuously at a relative growth rate of 1.17%, what will the population be in 8 years? Compute the answer to three significant digits.

9. POPULATION GROWTH In 2005 the population of Russia was 143 million and the population of Nigeria was 129 million. If the populations of Russia and Nigeria grow continuously at relative growth rates of −0.37% and 2.56%, respectively, in what year will Nigeria have a greater population than Russia?

10. POPULATION GROWTH In 2005 the population of Germany was 82 million and the population of Egypt was 78 million. If the populations of Germany and Egypt grow continuously at relative growth rates of 0% and 1.78%, respectively, in what year will Egypt have a greater population than Germany?

11. SPACE SCIENCE Radioactive isotopes, as well as solar cells, are used to supply power to space vehicles. The isotopes gradually lose power because of radioactive decay. On a particular space vehicle the nuclear energy source has a power output of P watts after t days of use as given by

$$P = 75e^{-0.0035t}$$

Graph this function for $0 \le t \le 100$.

12. EARTH SCIENCE The atmospheric pressure P, in pounds per square inch, decreases exponentially with altitude h, in miles above sea level, as given by

$$P = 14.7e^{-0.21h}$$

Graph this function for $0 \le h \le 10$.

13. MARINE BIOLOGY Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity I relative to depth d, in feet, for one of the clearest bodies of water in the world, the Sargasso Sea in the West Indies, can be approximated by

$$I = I_0e^{-0.00942d}$$

where I_0 is the intensity of light at the surface. To the nearest percent, what percentage of the surface light will reach a depth of
(A) 50 feet? (B) 100 feet?

14. MARINE BIOLOGY Refer to Problem 13. In some waters with a great deal of sediment, the photic zone may go down only 15 to 20 feet. In some murky harbors, the intensity of light d feet below the surface is given approximately by

$$I = I_0 e^{-0.23d}$$

What percentage of the surface light will reach a depth of
(A) 10 feet? (B) 20 feet?

15. AIDS EPIDEMIC The World Health Organization estimated that 39.4 million people worldwide were living with HIV in 2004. Assuming that number continues to increase at a relative growth rate of 3.2% compounded continuously, estimate the number of people living with HIV in
(A) 2010 (B) 2015

16. AIDS EPIDEMIC The World Health Organization estimated that there were 3.1 million deaths worldwide from HIV/AIDS during the year 2004. Assuming that number continues to increase at a relative growth rate of 4.3% compounded continuously, estimate the number of deaths from HIV/AIDS during the year
(A) 2008 (B) 2012

17. NEWTON'S LAW OF COOLING This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

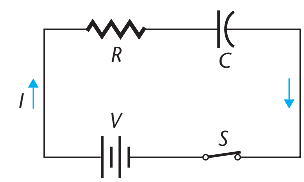
where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in the refrigerator to cool before a dinner party. If the temperature of in the refrigerator is kept at 40°F and $k = 0.4$, find the temperature of the wine, to the nearest degree, after 3 hours. (In Exercise 5-5 we will find out how to determine k.)

18. NEWTON'S LAW OF COOLING Refer to Problem 17. What is the temperature, to the nearest degree, of the wine after 5 hours in the refrigerator?

19. PHOTOGRAPHY An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered, and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time.* For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

Find the value that q approaches as t increases without bound and interpret.



20. MEDICINE An electronic heart pacemaker uses the same type of circuit as the flash unit in Problem 19, but it is designed so that the capacitor discharges 72 times a minute. For a particular pacemaker, the charge on the capacitor t seconds after it starts recharging is given by

$$q = 0.000\,008(1 - e^{-2t})$$

Find the value that q approaches as t increases without bound and interpret.

21. WILDLIFE MANAGEMENT A herd of 20 white-tailed deer is introduced to a coastal island where there had been no deer before. Their population is predicted to increase according to the logistic curve

$$N = \frac{100}{1 + 4e^{-0.14t}}$$

where N is the number of deer expected in the herd after t years.
(A) How many deer will be present after 2 years? After 6 years? Round answers to the nearest integer.
(B) How many years will it take for the herd to grow to 50 deer? Round answer to the nearest integer.
(C) Does N approach *a* limiting value as t increases without bound? Explain.

22. TRAINING A trainee is hired by a computer manufacturing company to learn to test a particular model of a personal computer after it comes off the assembly line. The learning curve for an average trainee is given by

$$N = \frac{200}{4 + 21e^{-0.1t}}$$

(A) How many computers can an average trainee be expected to test after 3 days of training? After 6 days? Round answers to the nearest integer.
(B) How many days will it take until an average trainee can test 30 computers per day? Round answer to the nearest integer.
(C) Does N approach a limiting value as t increases without bound? Explain.

Problems 23–26 require a graphing calculator or a computer that can calculate exponential and logistic regression models for a given data set.

23. DEPRECIATION Table 4 gives the market value of a minivan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the van. Estimate the value of the van 10 years after tis purchase. Round answers to the nearest dollar.

Table 4

x	Value ($)
1	12,575
2	9,455
3	8,115
4	6,845
5	5,225
6	4,485

Source: Kelley Blue Book

24. DEPRECIATION Table 5 gives the market value of a luxury sedan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the sedan. Estimate the value of the sedan 10 years after its purchase. Round answers to the nearest dollar.

Table 5

x	Value ($)
1	23,125
2	19,050
3	15,625
4	11,875
5	9,450
6	7,125

Source: Kelley Blue Book

25. NUCLEAR POWER Table 6 gives data on nuclear power generation by region for the years 1980–1999.

Table 6 Nuclear Power Generation

	(Billion Kilowatt-Hours)	
Year	North America	Central and South America
1980	287.0	2.2
1985	440.8	8.4
1990	649.0	9.0
1995	774.4	9.5
1998	750.2	10.3
1999	807.5	10.5

Source: U.S. Energy Information Administration

(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model $\left(y = \frac{c}{1 + ae^{-bx}}\right)$ for the generation of nuclear power in North America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in North America in 2010.

26. NUCLEAR POWER Refer to Table 6.
(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model $\left(y = \frac{c}{1 + ae^{-bx}}\right)$ for the generation of nuclear power in Central and South America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in Central and South America in 2010.

5-3 Logarithmic Functions

- Logarithmic Functions and Graphs
- From Logarithmic Form to Exponential Form, and Vice Versa
- Properties of Logarithmic Functions
- Common and Natural Logarithms
- Change of Base

In Section 5-3 we introduce the inverses of the exponential functions—the logarithmic functions—and study their properties and graphs.

Logarithmic Functions and Graphs

The exponential function $f(x) = b^x$, where $b > 0$, $b \neq 1$, is a one-to-one function, and therefore has an inverse. Its inverse, denoted $f^{-1}(x) = \log_b x$ (read "log to the base b of x"), is called the *logarithmic function with base b.* A point (x, y) lies on the graph of f^{-1} if and only if the point (y, x) lies on the graph of f; in other words,

$$y = \log_b x \text{ if and only if } x = b^y$$

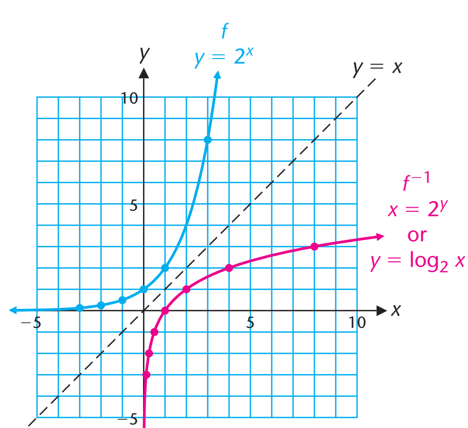
We can use this fact to deduce information about the logarithmic functions from our knowledge of exponential functions. For example, the graph of f^{-1} is the graph of f reflected in the line $y = x$; and the domain and range of f^{-1} are, respectively, the range and domain of f.

Consider the exponential function $f(x) = 2^x$ and its inverse $f^{-1}(x) = \log_2 x$. Figure 1 shows the graphs of both functions and a table of selected points on those graphs. Because

$$y = \log_2 x \text{ if and only if } x = 2^y$$

$\log_2 x$ is the exponent to which 2 must be raised to obtain x: $2^{\log_2 x} = 2^y = x$.

Figure 1 Logarithmic function with base 2.



DOMAIN of $f = (-\infty, \infty)$ = RANGE of f^{-1}
RANGE of $f = (0, \infty)$ = DOMAIN of f^{-1}

f: x	f: $y = 2^x$	f^{-1}: $x = 2^y$	f^{-1}: y
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

DEFINITION 1 Logarithmic Function

For $b > 0$, $b \neq 1$, the inverse of $f(x) = b^x$, denoted $f^{-1}(x) = \log_b x$, is the **logarithmic function with base *b*.**

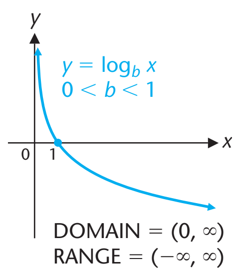
Logarithmic form		Exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

The log to the base b of x is the exponent to which b must be raised to obtain x.

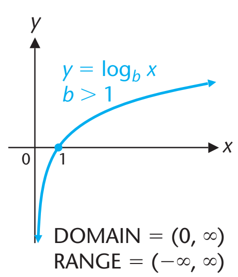
$$y = \log_{10} x \quad \text{is equivalent to} \quad x = 10^y$$

$$y = \log_e x \quad \text{is equivalent to} \quad x = e^y$$

Remember: A logarithm is an exponent.



(a)



(b)

Figure 2
Typical logarithmic graphs.

It is very important to remember that $y = \log_b x$ and $x = b^y$ define the same function, and as such can be used interchangeably.

Because the domain of an exponential function includes all real numbers and its range is the set of positive real numbers, the **domain** of a logarithmic function is the set of all positive real numbers and its **range** is the set of all real numbers. Thus, $\log_{10} 3$ is defined, but $\log_{10} 0$ and $\log_{10} (-5)$ are not defined. That is, 3 is a logarithmic domain value, but 0 and -5 are not. Typical logarithmic curves are shown in Figure 2.

The graphs of logarithmic functions have the properties stated in Theorem 1. These properties, suggested by the graphs in Figure 2, can be deduced from corresponding properties of the exponential functions.

THEOREM 1 Properties of Graphs of Logarithmic Functions

Let $f(x) = \log_b x$ be a logarithmic function, $b > 0$, $b \neq 1$. Then the graph of $f(x)$:

1. Is continuous on its domain $(0, \infty)$.
2. Has no sharp corners.
3. Passes through the point $(1, 0)$.
4. Lies to the right of the y axis, which is a vertical asymptote.
5. Increases as x increases if $b > 0$; decreases as x increases if $0 < b < 1$.
6. Intersects any horizontal line exactly once, so is one-to-one.

EXAMPLE 1 Transformations of Logarithmic Functions

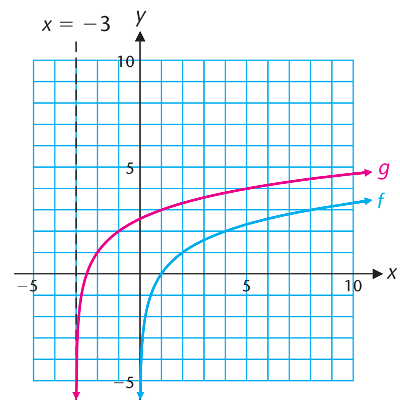
Let $g(x) = 1 + \log_2 (x + 3)$.

(A) Use transformations to explain how the graph of g is related to the graph of the logarithmic function $f(x) = \log_2 x$. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.

(B) Find the inverse of g.

SOLUTIONS

(A) The graph of g can be obtained from the graph of f by a horizontal translation (left 3 units) followed by a vertical translation (up 1 unit) (see Fig. 3). The graph of g is increasing. The domain of g is the set of real numbers x such that $x + 3 > 0$, namely $(-3, \infty)$. The line $x = -3$ is a vertical asymptote (indicated by the dashed line in Fig. 3).



› **Figure 3**
$f(x) = \log_2 x$, $g(x) = 1 + \log_2 (x + 3)$.

(B)

$$\begin{aligned} y &= 1 + \log_2 (x + 3) && \text{Subtract 1 from both sides.} \\ y - 1 &= \log_2 (x + 3) && \text{Write in exponential form.} \\ x + 3 &= 2^{y-1} && \text{Subtract 3 from both sides.} \\ x &= 2^{y-1} - 3 && \text{Interchange } x \text{ and } y. \\ y &= 2^{x-1} - 3 \end{aligned}$$

Therefore the inverse of g is $g^{-1}(x) = 2^{x-1} - 3$.

MATCHED PROBLEM 1

Let $g(x) = -2 + \log_2 (x - 4)$.

(A) Use transformations to explain how the graph of g is related to the graph of the logarithmic function $f(x) = \log_2 x$. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.

(B) Find the inverse of g.

››› EXPLORE-DISCUSS 1

For the exponential function $f = \{(x, y) \mid y = (\frac{2}{3})^x\}$, graph f and $y = x$ on the same coordinate system. Then sketch the graph of f^{-1}. Discuss the domains and ranges of f and its inverse. By what other name is f^{-1} known?

From Logarithmic Form to Exponential Form, and Vice Versa

We now look into the matter of converting logarithmic forms to equivalent exponential forms, and vice versa.

EXAMPLE 2 Logarithmic–Exponential Conversions

Change each logarithmic form to an equivalent exponential form.

(A) $\log_2 8 = 3$ (B) $\log_{25} 5 = \frac{1}{2}$ (C) $\log_2 (\frac{1}{4}) = -2$

SOLUTIONS

(A) $\log_2 8 = 3$ is equivalent to $8 = 2^3$.

(B) $\log_{25} 5 = \frac{1}{2}$ is equivalent to $5 = 25^{1/2}$.

(C) $\log_2 (\frac{1}{4}) = -2$ is equivalent to $\frac{1}{4} = 2^{-2}$.

MATCHED PROBLEM 2

Change each logarithmic form to an equivalent exponential form.

(A) $\log_3 27 = 3$ (B) $\log_{36} 6 = \frac{1}{2}$ (C) $\log_3 (\frac{1}{9}) = -2$

EXAMPLE 3 Logarithmic–Exponential Conversions

Change each exponential form to an equivalent logarithmic form.

(A) $49 = 7^2$ (B) $3 = \sqrt{9}$ (C) $\frac{1}{5} = 5^{-1}$

SOLUTIONS

(A) $49 = 7^2$ is equivalent to $\log_7 49 = 2$.

(B) $3 = \sqrt{9}$ is equivalent to $\log_9 3 = \frac{1}{2}$.

(C) $\frac{1}{5} = 5^{-1}$ is equivalent to $\log_5 (\frac{1}{5}) = -1$.

MATCHED PROBLEM 3

Change each exponential form to an equivalent logarithmic form.

(A) $64 = 4^3$ (B) $2 = \sqrt[3]{8}$ (C) $\frac{1}{16} = 4^{-2}$

To gain a little deeper understanding of logarithmic functions and their relationship to the exponential functions, we consider a few problems where we want to find x, b, or y in $y = \log_b x$, given the other two values. All values were chosen so that the problems can be solved without a calculator.

EXAMPLE 4 Solutions of the Equation $y = \log_b x$

Find x, b, or y as indicated.

(A) Find y: $y = \log_4 8$. (B) Find x: $\log_3 x = -2$.

(C) Find b: $\log_b 1{,}000 = 3$.

SOLUTIONS

(A) Write $y = \log_4 8$ in equivalent exponential form:

$$8 = 4^y \quad \text{Write each number to the same base 2.}$$
$$2^3 = 2^{2y} \quad \text{Recall that } b^m = b^n \text{ if and only if } m = n.$$
$$2y = 3 \quad \text{Divide both sides by 2.}$$
$$y = \tfrac{3}{2}$$

Thus, $\frac{3}{2} = \log_4 8$.

(B) Write $\log_3 x = -2$ in equivalent exponential form:

$$x = 3^{-2} \quad \text{Simplify.}$$
$$= \frac{1}{3^2} = \frac{1}{9}$$

Thus, $\log_3 (\frac{1}{9}) = -2$.

(C) Write $\log_b 1{,}000 = 3$ in equivalent exponential form:

$$1{,}000 = b^3 \quad \text{Write 1,000 as a third power.}$$
$$10^3 = b^3 \quad \text{Take cube roots.}$$
$$b = 10$$

Thus, $\log_{10} 1{,}000 = 3$.

MATCHED PROBLEM 4

Find x, b, or y as indicated.

(A) Find y: $y = \log_9 27$. (B) Find x: $\log_2 x = -3$.

(C) Find b: $\log_b 100 = 2$.

Properties of Logarithmic Functions

The familiar properties of exponential functions imply corresponding properties of logarithmic functions.

EXPLORE-DISCUSS 2

Discuss the connection between the exponential equation and the logarithmic equation, and explain why each equation is valid.

(A) $2^4\,2^7 = 2^{11}$; $\log_2 2^4 + \log_2 2^7 = \log_2 2^{11}$

(B) $2^{13}/2^5 = 2^8$; $\log_2 2^{13} - \log_2 2^5 = \log_2 2^8$

(C) $(2^6)^9 = 2^{54}$; $9 \log_2 2^6 = \log_2 2^{54}$

Several of the powerful and useful properties of logarithmic functions are listed in Theorem 2.

THEOREM 2 Properties of Logarithmic Functions

If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ if and only if $M = N$

The first two properties in Theorem 2 follow directly from the definition of a logarithmic function:

$$\log_b 1 = 0 \quad \text{because} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{because} \quad b^1 = b$$

The third and fourth properties are "inverse properties." They follow directly from the fact that exponential and logarithmic functions are inverses of each other. Recall from Section 3-6 that if f is one-to-one, then f^{-1} is a one-to-one function satisfying

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Applying these general properties to $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, we see that

$$\begin{aligned} f^{-1}(f(x)) &= x & \qquad f(f^{-1}(x)) &= x \\ \log_b (f(x)) &= x & b^{f^{-1}(x)} &= x \\ \log_b b^x &= x & b^{\log_b x} &= x \end{aligned}$$

Properties 5 to 7 enable us to convert multiplication into addition, division into subtraction, and power and root problems into multiplication. The proofs of these properties are based on properties of exponents. A sketch of a proof of the fifth property follows: To bring exponents into the proof, we let

$$u = \log_b M \qquad \text{and} \qquad v = \log_b N$$

and convert these to the equivalent exponential forms

$$M = b^u \qquad \text{and} \qquad N = b^v$$

Now, see if you can provide the reasons for each of the following steps:

$$\log_b MN = \log_b b^u b^v = \log_b b^{u+v} = u + v = \log_b M + \log_b N$$

The other properties are established in a similar manner (see Problems 125 and 126 in Exercise 5-3.)

Finally, the eighth property follows from the fact that logarithmic functions are one-to-one.

EXAMPLE 5 Using Logarithmic Properties

Simplify, using the properties in Theorem 2.

(A) $\log_e 1$ (B) $\log_{10} 10$ (C) $\log_e e^{2x+1}$

(D) $\log_{10} 0.01$ (E) $10^{\log_{10} 7}$ (F) $e^{\log_e x^2}$

SOLUTIONS

(A) $\log_e 1 = 0$ (B) $\log_{10} 10 = 1$

(C) $\log_e e^{2x+1} = 2x + 1$ (D) $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$

(E) $10^{\log_{10} 7} = 7$ (F) $e^{\log_e x^2} = x^2$

MATCHED PROBLEM 5

Simplify, using the properties in Theorem 2.

(A) $\log_{10} 10^{-5}$ (B) $\log_5 25$ (C) $\log_{10} 1$

(D) $\log_e e^{m+n}$ (E) $10^{\log_{10} 4}$ (F) $e^{\log_e (x^4+1)}$

Common and Natural Logarithms

John Napier (1550–1617) is credited with the invention of logarithms, which evolved out of an interest in reducing the computational strain in research in astronomy. This new computational tool was immediately accepted by the scientific world. Now, with the availability of inexpensive calculators, logarithms have lost most of their importance as a computational device. However, the logarithmic concept has been greatly generalized since its conception, and logarithmic functions are used widely in both theoretical and applied sciences.

Of all possible logarithmic bases, the base e and the base 10 are used almost exclusively. To use logarithms in certain practical problems, we need to be able to approximate the logarithm of any positive number to either base 10 or base e. And conversely, if we are given the logarithm of a number to base 10 or base e, we need to be able to approximate the number. Historically, tables were used for this purpose, but now calculators are used because they are faster and can find far more values than any table can possibly include.

Common logarithms, also called **Briggsian logarithms,** are logarithms with base 10. **Natural logarithms,** also called **Napierian logarithms,** are logarithms with base e. Most calculators have a function key labeled "log" and a function key labeled "ln." The former represents the common logarithmic function and the latter the natural logarithmic function. In fact, "log" and "ln" are both used extensively in mathematical literature, and whenever you see either used in this book without a base indicated, they should be interpreted as in the box.

LOGARITHMIC FUNCTIONS

$y = \log x = \log_{10} x$ Common logarithmic function

$y = \ln x = \log_e x$ Natural logarithmic function

EXPLORE-DISCUSS 3

(A) Sketch the graph of $y = 10^x$, $y = \log x$, and $y = x$ in the same coordinate system and state the domain and range of the common logarithmic function.

(B) Sketch the graph of $y = e^x$, $y = \ln x$, and $y = x$ in the same coordinate system and state the domain and range of the natural logarithmic function.

EXAMPLE 6 Calculator Evaluation of Logarithms

Use a calculator to evaluate each to six decimal places.

(A) $\log 3{,}184$ (B) $\ln 0.000\,349$ (C) $\log(-3.24)$

SOLUTIONS

(A) $\log 3{,}184 = 3.502\,973$

(B) $\ln 0.000\,349 = -7.960\,439$

(C) $\log(-3.24) =$ Error

Why is an error indicated in part C? Because -3.24 is not in the domain of the log function. [*Note:* Calculators display error messages in various ways. Some calculators use a more advanced definition of logarithmic functions that involves complex numbers. They will display an ordered pair, representing a complex number, as the value of $\log(-3.24)$, rather than an error message. You should interpret such a display as indicating that the number entered is not in the domain of the logarithmic function as we have defined it.]

MATCHED PROBLEM 6

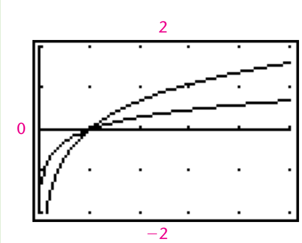
Use a calculator to evaluate each to six decimal places.

(A) $\log 0.013\,529$ (B) $\ln 28.693\,28$ (C) $\ln(-0.438)$

When working with common and natural logarithms, we follow the common practice of using the equal sign "=" where it might be more appropriate to use the approximately equal sign "≈." No harm is done as long as we keep in mind that in a statement such as $\log 3.184 = 0.503$, the number on the right is only assumed accurate to three decimal places and is not exact.

››› EXPLORE-DISCUSS 4

Graphs of the functions $f(x) = \log x$ and $g(x) = \ln x$ are shown in the graphing utility display of Figure 4. Which graph belongs to which function? It appears from the display that one of the functions may be a constant multiple of the other. Is that true? Find and discuss the evidence for your answer.



› Figure 4

EXAMPLE 7

Calculator Evaluation of Logarithms

Use a calculator to evaluate each expression to three decimal places.

(A) $\dfrac{\log 2}{\log 1.1}$ (B) $\log \dfrac{2}{1.1}$ (C) $\log 2 - \log 1.1$

SOLUTIONS

(A) $\dfrac{\log 2}{\log 1.1} = 7.273$ (B) $\log \dfrac{2}{1.1} = 0.260$

(C) $\log 2 - \log 1.1 = 0.260$. Note that $\dfrac{\log 2}{\log 1.1} \neq \log 2 - \log 1.1$, but

$\log \dfrac{2}{1.1} = \log 2 - \log 1.1$ (see Theorem 1).

MATCHED PROBLEM 7

Use a calculator to evaluate each to three decimal places.

(A) $\dfrac{\ln 3}{\ln 1.08}$ (B) $\ln \dfrac{3}{1.08}$ (C) $\ln 3 - \ln 1.08$

We now turn to the second problem: Given the logarithm of a number, find the number. To solve this problem, we make direct use of the logarithmic–exponential relationships.

LOGARITHMIC–EXPONENTIAL RELATIONSHIPS

$\log x = y$	is equivalent to	$x = 10^y$
$\ln x = y$	is equivalent to	$x = e^y$

EXAMPLE 8

Solving $\log_b x = y$ for x

Find x to three significant digits, given the indicated logarithms.

(A) $\log x = -9.315$ (B) $\ln x = 2.386$

SOLUTIONS

(A) $\log x = -9.315$ Change to exponential form (Definition 1).

$x = 10^{-9.315}$ Calculate to three significant digits.

$= 4.84 \times 10^{-10}$

Notice that the answer is displayed in scientific notation in the calculator.

(B) $\ln x = 2.386$ — Change to exponential form (Definition 1).

$$x = e^{2.386}$$ Calculate to three significant digits.

$$= 10.9$$

MATCHED PROBLEM 8

Find x to four significant digits, given the indicated logarithms.

(A) $\ln x = -5.062$ (B) $\log x = 12.0821$

Technology Connections

Example 8 was solved algebraically using the logarithmic–exponential relationships. Use the intersection routine on a graphing utility to solve this problem graphically. Discuss the relative merits of the two approaches.

› Change of Base

How would you find the logarithm of a positive number to a base other than 10 or e? For example, how would you find $\log_3 5.2$? In Example 9 we evaluate this logarithm using a direct process. Then we develop a change-of-base formula to find such logarithms in general. You may find it easier to remember the process than the formula.

EXAMPLE 9 Evaluating a Base 3 Logarithm

Evaluate $\log_3 5.2$ to four decimal places.

SOLUTIONS

Let $y = \log_3 5.2$ and proceed as follows:

$$\log_3 5.2 = y$$ Change to exponential form.

$$5.2 = 3^y$$ Take the natural log (or common log) of each side.

$$\ln 5.2 = \ln 3^y$$ $\log_b M^p = p \log_b M$

$$= y \ln 3$$ Divide both sides by ln 3.

$$y = \frac{\ln 5.2}{\ln 3}$$

Replace y with $\log_3 5.2$ from the first step, and use a calculator to evaluate the right side:

$$\log_3 5.2 = \frac{\ln 5.2}{\ln 3} = 1.5007$$

MATCHED PROBLEM 9

Evaluate $\log_{0.5} 0.0372$ to four decimal places.

To develop a change-of-base formula for arbitrary positive bases, with neither base equal to 1, we proceed as in Example 9. Let $y = \log_b N$, where N and b are positive and $b \neq 1$. Then

$$\begin{aligned} \log_b N &= y && \text{Write in exponential form.} \\ N &= b^y && \text{Take the log of each side to another positive base } a, a \neq 1. \\ \log_a N &= \log_a b^y && \log_a M^p = p \log_a M \\ \log_a N &= y \log_a b && \text{Divide both sides by } \log_a b. \\ y &= \frac{\log_a N}{\log_a b} \end{aligned}$$

Replacing y with $\log_b N$ from the first step, we obtain the **change-of-base formula:**

$$\log_b N = \frac{\log_a N}{\log_a b}$$

In words, this formula states that the logarithm of a number to a given base is the logarithm of that number to a new base divided by the logarithm of the old base to the new base. In practice, we usually choose either e or 10 for the new base so that a calculator can be used to evaluate the necessary logarithms:

$$\log_b N = \frac{\ln N}{\ln b} \quad \text{or} \quad \log_b N = \frac{\log N}{\log b}$$

We used the first of these options in Example 9.

››› EXPLORE-DISCUSS 5

If b is any positive real number different from 1, the change-of-base formula implies that the function $y = \log_b x$ is a constant multiple of the natural logarithmic function; that is, $\log_b x = k \ln x$ for some k.

(A) Graph the functions $y = \ln x$, $y = 2 \ln x$, $y = 0.5 \ln x$, and $y = -3 \ln x$.

(B) Write each function of part A in the form $y = \log_b x$ by finding the base b to two decimal places.

(C) Is every exponential function $y = b^x$ a constant multiple of $y = e^x$? Explain.

››› CAUTION ›››

We conclude this section by noting two common errors:

1. $\dfrac{\log_b M}{\log_b N} \neq \log_b M - \log_b N$

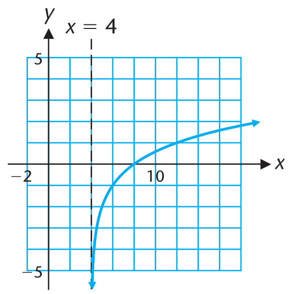
$\log_b M - \log_b N = \log_b \dfrac{M}{N}$; $\dfrac{\log_b M}{\log_b N}$ cannot be simplified.

2. $\log_b (M + N) \neq \log_b M + \log_b N$

$\log_b M + \log_b N = \log_b MN$; $\log_b (M + N)$ cannot be simplified.

ANSWERS TO MATCHED PROBLEMS

1. (A) The graph of g is the same as the graph of f shifted 4 units to the right and 2 units down; g is increasing; domain: $(4, \infty)$; vertical asymptote: $x = 4$



(B) $g^{-1}(x) = 4 + 2^{x+2}$

2. (A) $27 = 3^3$ (B) $6 = 36^{1/2}$ (C) $\frac{1}{9} = 3^{-2}$

3. (A) $\log_4 64 = 3$ (B) $\log_8 2 = \frac{1}{3}$ (C) $\log_4 (\frac{1}{16}) = -2$

4. (A) $y = \frac{3}{2}$ (B) $x = \frac{1}{8}$ (C) $b = 10$

5. (A) -5 (B) 2 (C) 0 (D) $m + n$ (E) 4 (F) $x^4 + 1$

6. (A) $-1.868\,734$ (B) $3.356\,663$ (C) Not possible

7. (A) 14.275 (B) 1.022 (C) 1.022

8. (A) $x = 0.006\,333$ (B) $x = 1.208 \times 10^{12}$

9. 4.7486

5-3 Exercises

Rewrite Problems 1–10 in equivalent exponential form.

1. $\log_3 81 = 4$

2. $\log_5 125 = 3$

3. $\log_{10} 0.001 = -3$

4. $\log_{10} 1{,}000 = 3$

5. $\log_{81} 3 = \frac{1}{4}$

6. $\log_4 2 = \frac{1}{2}$

7. $\log_6 \frac{1}{36} = -2$

8. $\log_2 \frac{1}{64} = -6$

9. $\log_{1/2} 16 = -4$

10. $\log_{1/3} 27 = -3$

Rewrite Problems 11–20 in equivalent logarithmic form.

11.

12.

13.

14.

15. $\frac{1}{2} = 32^{-1}$

16. $\frac{1}{8} = 2^{-3}$

17. $7 = 49^{1/2}$

18. $4 = 64^{1/3}$

19. $(\frac{2}{3})^3 = \frac{8}{27}$

20. $(\frac{5}{2})^{-2} = 0.16$

In Problems 21–40, simplify each expression using Theorem 2.

21. $\log_{16} 1$

22. $\log_{25} 1$

23. $\log_{0.5} 0.5$

24. $\log_7 7$

25. $\log_e e^4$

26. $\log_{10} 10^5$

27. $\log_{10} 0.01$

28. $\log_{10} 100$

29. $\log_3 27$

30. $\log_4 256$

31. $\log_{1/2} 2$

32. $\log_{1/5} \frac{1}{25}$

33. $e^{\log_e 5}$

34. $e^{\log_e 10}$

35. $\log_5 \sqrt[3]{5}$

36. $\log_2 \sqrt{8}$

37. $e^{\log_e \sqrt{x}}$

38. $e^{\log_e (x-1)}$

39. $e^{2 \log_e x}$

40. $10^{-3 \log_{10} u}$

In Problems 41–48, evaluate to four decimal places.

41. $\log 49{,}236$

42. $\log 691{,}450$

43. $\ln 54.081$

44. $\ln 19.722$

45. $\log_7 13$

46. $\log_9 78$

47. $\log_5 120.24$

48. $\log_{17} 304.66$

In Problems 49–56, evaluate x to four significant digits, given:

49. $\log x = 5.3027$

50. $\log x = 1.9168$

51. $\log x = -3.1773$

52. $\log x = -2.0411$

53. $\ln x = 3.8655$

54. $\ln x = 5.0884$

55. $\ln x = -0.3916$

56. $\ln x = -4.1083$

Find x, y, or b, as indicated in Problems 57–74.

57. $\log_2 x = 2$

58. $\log_3 x = 3$

59. $\log_4 16 = y$

60. $\log_8 64 = y$

61. $\log_b 16 = 2$

62. $\log_b 10^{-3} = -3$

63. $\log_b 1 = 0$

64. $\log_b b = 1$

In Problems 95–98, evaluate to five significant digits.

95. $\log (5.3147 \times 10^{12})$

96. $\log (2.0991 \times 10^{17})$

97. $\ln (6.7917 \times 10^{-12})$

98. $\ln (4.0304 \times 10^{-8})$

In Problems 99–106, use transformations to explain how the graph of g is related to the graph of the given logarithmic function f. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.

99. $g(x) = 3 + \log_2 x$; $f(x) = \log_2 x$

100. $g(x) = -4 + \log_3 x$; $f(x) = \log_3 x$

101. $g(x) = \log_{1/3} (x - 2)$; $f(x) = \log_{1/3} x$

102. $g(x) = \log_{1/2} (x + 3)$; $f(x) = \log_{1/2} x$

103. $g(x) = -1 - \log x$; $f(x) = \log x$

104. $g(x) = 2 - \log x$; $f(x) = \log x$

105. $g(x) = 5 - 3 \ln x$; $f(x) = \ln x$

106. $g(x) = -3 - 2 \ln x$; $f(x) = \ln x$

In Problems 107–114, find f^{-1}.

107. $f(x) = \log_5 x$

108. $f(x) = \log_{1/3} x$

109. $f(x) = 4 \log_3 (x + 3)$

110. $f(x) = 2 \log_2 (x - 5)$

111. $f(x) = 4 - 2 \log (x + 1)$

112. $f(x) = -3 + 5 \log (x + 2)$

113. $f(x) = -1 + \frac{1}{2} \ln (x - 5)$

114. $f(x) = 6 - \frac{2}{3} \ln (x - 1)$

115. Let $f(x) = \log_3 (2 - x)$.
(A) Find f^{-1}.
(B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

116. Let $f(x) = \log_2 (-3 - x)$.
(A) Find f^{-1}.
(B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

117. Find the fallacy.

$$\begin{aligned}
1 &< 3 && \text{Divide both sides by 27.}\\
\tfrac{1}{27} &< \tfrac{3}{27}\\
\tfrac{1}{27} &< \tfrac{1}{9}\\
(\tfrac{1}{3})^3 &< (\tfrac{1}{3})^2\\
\log (\tfrac{1}{3})^3 &< \log (\tfrac{1}{3})^2\\
3 \log \tfrac{1}{3} &< 2 \log \tfrac{1}{3} && \text{Divide both sides by } \log \tfrac{1}{3}.\\
3 &< 2
\end{aligned}$$

118. Find the fallacy.

$$\begin{aligned}
3 &> 2 && \text{Multiply both sides by } \log \tfrac{1}{2}.\\
3 \log \tfrac{1}{2} &> 2 \log \tfrac{1}{2}\\
\log (\tfrac{1}{2})^3 &> \log (\tfrac{1}{2})^2\\
(\tfrac{1}{2})^3 &> (\tfrac{1}{2})^2\\
\tfrac{1}{8} &> \tfrac{1}{4} && \text{Multiply both sides by 8.}\\
1 &> 2
\end{aligned}$$

119. The function $f(x) = \log x$ increases extremely slowly as $x \to \infty$, but the composite function $g(x) = \log (\log x)$ increases still more slowly.
(A) Illustrate this fact by computing the values of both functions for several large values of x.
(B) Determine the domain and range of the function g.
(C) Discuss the graphs of both functions.

120. The function $f(x) = \ln x$ increases extremely slowly as $x \to \infty$, but the composite function $g(x) = \ln(\ln x)$ increases still more slowly.
(A) Illustrate this fact by computing the values of both functions for several large values of x.
(B) Determine the domain and range of the function g.
(C) Discuss the graphs of both functions.

The polynomials in Problems 121–124, called ***Taylor polynomials,*** *can be used to approximate the function $g(x) = \ln (1 + x)$. To illustrate this approximation graphically, in each problem, graph $g(x) = \ln (1 + x)$ and the indicated polynomial in the same viewing window, $-1 \le x \le 3$ and $-2 \le y \le 2$.*

121. $P_1(x) = x - \frac{1}{2}x^2$

122. $P_2(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

123. $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

124. $P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

125. Prove that $\log_b (M/N) = \log_b M - \log_b N$ under the hypotheses of Theorem 2.

126. Prove that $\log_b M^p = p \log_b M$ under the hypotheses of Theorem 2.

5-4 Logarithmic Models

- Logarithmic Scales
- Data Analysis and Regression

In Section 5-4 we study the logarithmic scales that are used to compare intensities of sounds, magnitudes of earthquakes, and the brightness of stars. We construct logarithmic models using regression techniques.

Logarithmic Scales

SOUND INTENSITY The human ear is able to hear sound over an incredible range of intensities. The loudest sound a healthy person can hear without damage to the eardrum has an intensity 1 trillion (1,000,000,000,000) times that of the softest sound a person can hear. Working directly with numbers over such a wide range is very cumbersome. Because the logarithm, with base greater than 1, of a number increases much more slowly than the number itself, logarithms are often used to create more convenient compressed scales. The decibel scale for sound intensity is an example of such a scale. The **decibel,** named after the inventor of the telephone, Alexander Graham Bell (1847–1922), is defined as follows:

$$D = 10 \log \frac{I}{I_0} \qquad \text{Decibel scale} \qquad (1)$$

where D is the **decibel level** of the sound, I is the **intensity** of the sound measured in watts per square meter (W/m^2), and I_0 is the intensity of the least audible sound that an average healthy young person can hear. The latter is standardized to be $I_0 = 10^{-12}$ watts per square meter. Table 1 lists some typical sound intensities from familiar sources.

Table 1 Typical Sound Intensities

Sound Intensity (W/m^2)	Sound
1.0×10^{-12}	Threshold of hearing
5.2×10^{-10}	Whisper
3.2×10^{-6}	Normal conversation
8.5×10^{-4}	Heavy traffic
3.2×10^{-3}	Jackhammer
1.0×10^{0}	Threshold of pain
8.3×10^{2}	Jet plane with afterburner

EXAMPLE 1 Sound Intensity

Find the number of decibels from a whisper with sound intensity 5.20×10^{-10} watts per square meter. Compute the answer to two decimal places.

SOLUTION

We use the decibel formula (1):

$$
\begin{aligned}
D &= 10 \log \frac{I}{I_0} && \text{Let } I = 5.20 \times 10^{-10} \text{ and } I_0 = 10^{-12}. \\
&= 10 \log \frac{5.2 \times 10^{-10}}{10^{-12}} && \text{Simplify.} \\
&= 10 \log 520 && \text{Calculate to two decimal places.} \\
&= 27.16 \text{ decibels}
\end{aligned}
$$

MATCHED PROBLEM 1

Find the number of decibels from a jackhammer with sound intensity 3.2×10^{-3} watts per square meter. Compute the answer to two decimal places.

››› EXPLORE-DISCUSS 1

Imagine using a large sheet of graph paper, ruled with horizontal and vertical lines $\frac{1}{8}$-inch apart, to plot the sound intensities of Table 1 on the x axis and the corresponding decibel levels on the y axis. Suppose that each $\frac{1}{8}$-inch unit on the x axis represents the intensity of the least audible sound (10^{-12} W/m^2), and each $\frac{1}{8}$-inch unit on the y axis represents 1 decibel. If the point corresponding to a jet plane with afterburner is plotted on the graph paper, how far is it from the x axis? From the y axis? (Give the first answer in inches and the second in miles!) Discuss.

EARTHQUAKE INTENSITY The energy released by the largest earthquake recorded, measured in joules, is about 100 billion (100,000,000,000) times the energy released by a small earthquake that is barely felt. Over the past 150 years several people from various countries have devised different types of measures of earthquake magnitudes so that their severity could be easily compared. In 1935 the California seismologist Charles Richter devised a logarithmic scale that bears his name and is still widely used in the United States. The **magnitude** M on the **Richter scale*** is given as follows:

$$M = \frac{2}{3} \log \frac{E}{E_0} \qquad \text{Richter scale} \tag{2}$$

*Originally, Richter defined the magnitude of an earthquake in terms of logarithms of the maximum seismic wave amplitude, in thousandths of a millimeter, measured on a standard seismograph. Formula (2) gives essentially the same magnitude that Richter obtained for a given earthquake but in terms of logarithms of the energy released by the earthquake.

where E is the energy released by the earthquake, measured in joules, and E_0 is the energy released by a very small reference earthquake, which has been standardized to be

$$E_0 = 10^{4.40} \text{ joules}$$

The destructive power of earthquakes relative to magnitudes on the Richter scale is indicated in Table 2.

Table 2 The Richter Scale

Magnitude on Richter Scale	Destructive Power
$M < 4.5$	Small
$4.5 < M < 5.5$	Moderate
$5.5 < M < 6.5$	Large
$6.5 < M < 7.5$	Major
$7.5 < M$	Greatest

EXAMPLE 2

Earthquake Intensity

The 1906 San Francisco earthquake released approximately 5.96×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to two decimal places.

SOLUTION

We use the magnitude formula (2):

$$M = \frac{2}{3} \log \frac{E}{E_0} \qquad \text{Let } E = 5.96 \times 10^{16} \text{ and } E_0 = 10^{4.40}.$$

$$= \frac{2}{3} \log \frac{5.96 \times 10^{16}}{10^{4.40}} \qquad \text{Calculate to two decimal places.}$$

$$= 8.25$$

MATCHED PROBLEM 2

The 1985 earthquake in central Chile released approximately 1.26×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to two decimal places.

EXAMPLE 3 Earthquake Intensity

If the energy release of one earthquake is 1,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

SOLUTION

Let

$$M_1 = \frac{2}{3} \log \frac{E_1}{E_0} \quad \text{and} \quad M_2 = \frac{2}{3} \log \frac{E_2}{E_0}$$

be the Richter equations for the smaller and larger earthquakes, respectively. Substituting $E_2 = 1{,}000E_1$ into the second equation, we obtain

$$\begin{aligned} M_2 &= \frac{2}{3} \log \frac{1{,}000E_1}{E_0} && \text{log } MN = \log M + \log N \\ &= \frac{2}{3}\left(\log 10^3 + \log \frac{E_1}{E_0}\right) && \log 10^x = x \\ &= \frac{2}{3}\left(3 + \log \frac{E_1}{E_0}\right) && \text{Distributive property} \\ &= \frac{2}{3}(3) + \frac{2}{3} \log \frac{E_1}{E_0} && \text{Simplify.} \\ &= 2 + M_1 \end{aligned}$$

Thus, an earthquake with 1,000 times the energy of another has a Richter scale reading of 2 more than the other.

MATCHED PROBLEM 3

If the energy release of one earthquake is 10,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

ROCKET FLIGHT The theory of rocket flight uses advanced mathematics and physics to show that the **velocity** v of a rocket at burnout (depletion of fuel supply) is given by

$$v = c \ln \frac{W_t}{W_b} \qquad \text{Rocket equation} \qquad (3)$$

where c is the exhaust velocity of the rocket engine, W_t is the takeoff weight (fuel, structure, and payload), and W_b is the burnout weight (structure and payload).

Because of the Earth's atmospheric resistance, a launch vehicle velocity of at least 9.0 kilometers per second is required to achieve the minimum altitude needed for a stable orbit. It is clear that to increase velocity v, either the weight ratio W_t/W_b must be increased or the exhaust velocity c must be increased. The weight ratio can be increased by the use of solid fuels, and the exhaust velocity can be increased by improving the fuels, solid or liquid.

EXAMPLE 4

Rocket Flight Theory

A typical single-stage, solid-fuel rocket may have a weight ratio $W_t/W_b = 18.7$ and an exhaust velocity $c = 2.38$ kilometers per second. Would this rocket reach a launch velocity of 9.0 kilometers per second?

SOLUTION

We use the rocket equation (3):

$$v = c \ln \frac{W_t}{W_b}$$ Let $c = 2.38$ and $W_t/W_b = 18.7$.

$$= 2.38 \ln 18.7$$ Calculate to two decimal places.

$$= 6.97 \text{ kilometers per second}$$

The velocity of the launch vehicle is far short of the 9.0 kilometers per second required to achieve orbit. This is why multiple-stage launchers are used—the deadweight from a preceding stage can be jettisoned into the ocean when the next stage takes over.

MATCHED PROBLEM 4

A launch vehicle using liquid fuel, such as a mixture of liquid hydrogen and liquid oxygen, can produce an exhaust velocity of $c = 4.7$ kilometers per second. However, the weight ratio W_t/W_b must be low—around 5.5 for some vehicles—because of the increased structural weight to accommodate the liquid fuel. How much more or less than the 9.0 kilometers per second required to reach orbit will be achieved by this vehicle?

Data Analysis and Regression

We use logarithmic regression to fit a function of the form $y = a + b \ln x$ to a set of data points, making use of the techniques introduced earlier for linear, quadratic, exponential, and logistic functions.

EXAMPLE 5 Home Ownership Rates

The U.S. Census Bureau published the data in Table 3 on home ownership rates.

Table 3 Home Ownership Rates

Year	Home Ownership Rate (%)
1940	43.6
1950	55.0
1960	61.9
1970	62.9
1980	64.4
1990	64.2
2000	67.4

A logarithmic model for the data is given by

$$R = -36.7 + 23.0 \ln t$$

where R is the home ownership rate and t is time in years with $t = 0$ representing 1900.

(A) Use the model to predict the home ownership rate in 2010.

(B) Compare the actual home ownership rate in 1950 to the rate given by the model.

SOLUTIONS

(A) The year 2010 is represented by $t = 110$. Evaluating

$$R = -36.7 + 23.0 \ln t$$

at $t = 110$ predicts a home ownership rate of 71.4% in 2010.

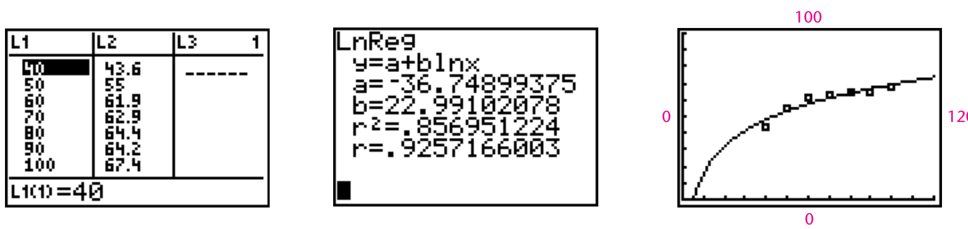
(B) The year 1950 is represented by $t = 50$. Evaluating

$$R = -36.7 + 23.0 \ln t$$

at $t = 50$ gives a home ownership rate of 53.3% in 1950. The actual home ownership rate in 1950 was 55%, approximately 1.7% greater than the number given by the model.

Technology Connections

Figure 1 shows the details of constructing the logarithmic model of Example 5 on a graphing calculator.



(a) Entering the data (b) Finding the model (c) Graphing the data and the model

› **Figure 1**

MATCHED PROBLEM 5

Refer to Example 5. The home ownership rate in 1995 was 64.7%. If this data is added to Table 3, a logarithmic model for the expanded data is given by

$$R = -31.5 + 21.7 \ln t$$

where R is the home ownership rate and t is time in years with $t = 0$ representing 1900.

(A) Use the model to predict the home ownership rate in 2010.

(B) Compare the actual home ownership rate in 1950 to the rate given by the model.

ANSWERS TO MATCHED PROBLEMS

1. 95.05 decibels **2.** 7.80 **3.** 2.67 **4.** 1 kilometer per second less
5. (A) 70.5% (B) The actual rate was 1.6% greater than the rate given by the model.

5-4 Exercises

APPLICATIONS

1. SOUND What is the decibel level of
(A) The threshold of hearing, 1.0×10^{-12} watts per square meter?
(B) The threshold of pain, 1.0 watt per square meter?

Compute answers to two significant digits.

2. SOUND What is the decibel level of
(A) A normal conversation, 3.2×10^{-6} watts per square meter?
(B) A jet plane with an afterburner, 8.3×10^{2} watts per square meter?

Compute answers to two significant digits.

3. SOUND If the intensity of a sound from one source is 1,000 times that of another, how much more is the decibel level of the louder sound than the quieter one?

4. SOUND If the intensity of a sound from one source is 10,000 times that of another, how much more is the decibel level of the louder sound than the quieter one?

5. EARTHQUAKES The strongest recorded earthquake to date was in Colombia in 1906, with an energy release of 1.99×10^{17} joules. What was its magnitude on the Richter scale? Compute the answer to one decimal place.

6. EARTHQUAKES Anchorage, Alaska, had a major earthquake in 1964 that released 7.08×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to one decimal place.

★★ **7.** EARTHQUAKES The 1933 Long Beach, California, earthquake had a Richter scale reading of 6.3, and the 1964 Anchorage, Alaska, earthquake had a Richter scale reading of 8.3. How many times more powerful was the Anchorage earthquake than the Long Beach earthquake?

★★ **8.** EARTHQUAKES Generally, an earthquake requires a magnitude of over 5.6 on the Richter scale to inflict serious damage. How many times more powerful than this was the great 1906 Colombia earthquake, which registered a magnitude of 8.6 on the Richter scale?

9. SPACE VEHICLES A new solid-fuel rocket has a weight ratio $W_t/W_b = 19.8$ and an exhaust velocity $c = 2.57$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places.

10. SPACE VEHICLES A liquid-fuel rocket has a weight ratio $W_t/W_b = 6.2$ and an exhaust velocity $c = 5.2$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places.

11. CHEMISTRY The hydrogen ion concentration of a substance is related to its acidity and basicity. Because hydrogen ion concentrations vary over a very wide range, logarithms are used to create a compressed **pH scale,** which is defined as follows:

$$\text{pH} = -\log\,[\text{H}^+]$$

where $[\text{H}^+]$ is the hydrogen ion concentration, in moles per liter. Pure water has a pH of 7, which means it is neutral. Substances with a pH less than 7 are acidic, and those with a pH greater than 7 are basic. Compute the pH of each substance listed, given the indicated hydrogen ion concentration.
(A) Seawater, 4.63×10^{-9}
(B) Vinegar, 9.32×10^{-4}

Also, indicate whether each substance is acidic or basic. Compute answers to one decimal place.

12. CHEMISTRY Refer to Problem 11. Compute the pH of each substance, given the indicated hydrogen ion concentration. Also, indicate whether it is acidic or basic. Compute answers to one decimal place.
(A) Milk, 2.83×10^{-7}
(B) Garden mulch, 3.78×10^{-6}

★**13.** ECOLOGY Refer to Problem 11. Many lakes in Canada and the United States will no longer sustain some forms of wildlife because of the increase in acidity of the water from acid rain and snow caused by sulfur dioxide emissions from industry. If the pH of a sample of rainwater is 5.2, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits.

★**14.** ECOLOGY Refer to Problem 11. If normal rainwater has a pH of 5.7, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits.

★★**15.** ASTRONOMY The brightness of stars is expressed in terms of magnitudes on a numerical scale that increases as the brightness decreases. The magnitude m is given by the formula

$$m = 6 - 2.5 \log \frac{L}{L_0}$$

where L is the light flux of the star and L_0 is the light flux of the dimmest stars visible to the naked eye.
(A) What is the magnitude of the dimmest stars visible to the naked eye?
(B) How many times brighter is a star of magnitude 1 than a star of magnitude 6?

16. ASTRONOMY An optical instrument is required to observe stars beyond the sixth magnitude, the limit of ordinary vision. However, even optical instruments have their limitations. The limiting magnitude L of any optical telescope with lens diameter D, in inches, is given by

$$L = 8.8 + 5.1 \log D$$

(A) Find the limiting magnitude for a homemade 6-inch reflecting telescope.
(B) Find the diameter of a lens that would have a limiting magnitude of 20.6.

Compute answers to three significant digits.

Problems 17 and 18 require a graphing calculator or a computer that can calculate a logarithmic regression model for a given data set.

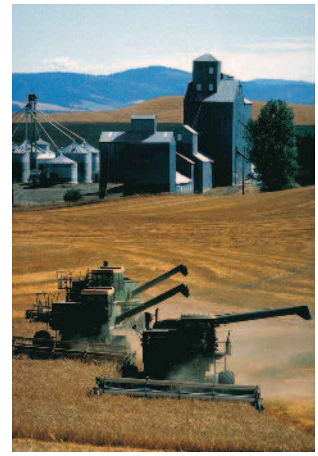
17. AGRICULTURE Table 4 shows the yield (bushels per acre) and the total production (millions of bushels) for corn in the United States for selected years since 1950. Let x represent years since 1900.

Table 4 United States Corn Production

Year	x	Yield (Bushels per Acre)	Total Production (Million Bushels)
1950	50	37.6	2,782
1960	60	55.6	3,479
1970	70	81.4	4,802
1980	80	97.7	6,867
1990	90	115.6	7,802
2000	100	137.0	9,915

Source: U.S. Department of Agriculture.

(A) Find a logarithmic regression model ($y = a + b \ln x$) for the yield. Estimate (to one decimal place) the yield in 2003 and in 2010.
(B) The actual yield in 2003 was 142 bushels per acre. How does this compare with the estimated yield in part A? What effect with this additional 2003 information have on the estimate for 2010? Explain.



18. AGRICULTURE Refer to Table 4.
(A) Find a logarithmic regression model ($y = a + b \ln x$) for the total production. Estimate (to the nearest million) the production in 2003 and in 2010.
(B) The actual production in 2003 was 10,114 million bushels. How does this compare with the estimated production in part A? What effect will this 2003 production information have on the estimate for 2010? Explain.

5-5 Exponential and Logarithmic Equations

- Exponential Equations
- Logarithmic Equations

Equations involving exponential and logarithmic functions, for example

$$2^{3x-2} = 5 \qquad \text{and} \qquad \log(x+3) + \log x = 1$$

are called **exponential** and **logarithmic equations,** respectively. We solve such equations to find the x intercepts of a function, or more generally, to find where the graphs of two functions intersect. Logarithmic properties play a central role in the solution of both exponential and logarithmic equations.

Exponential Equations

Examples 1–4 illustrate the use of logarithmic properties in solving exponential equations.

EXAMPLE 1 Finding x Intercepts

Find the x intercept(s) of $f(x) = 2^{3x-2} - 5$ to four decimal places.

SOLUTION

The x intercepts are the solutions of the equation $2^{3x-2} - 5 = 0$, or equivalently, $2^{3x-2} = 5$. How can we get the variable x out of the exponent? Use logs!

$$
\begin{aligned}
2^{3x-2} &= 5 && \text{Take the common or natural log of both sides.}\\
\log 2^{3x-2} &= \log 5 && \text{Use } \log_b N^p = p \log_b N \text{ to get } 3x - 2 \text{ out of the exponent position.}\\
(3x-2)\log 2 &= \log 5 && \text{Divide both sides by log 2.}\\
3x - 2 &= \frac{\log 5}{\log 2} && \text{Add 2 to both sides.}\\
3x &= 2 + \frac{\log 5}{\log 2} && \text{Divide both sides by 3.}\\
x &= \frac{1}{3}\left(2 + \frac{\log 5}{\log 2}\right) && \textit{Remember: } \frac{\log 5}{\log 2} \neq \log 5 - \log 2.\\
&= 1.4406 && \text{To four decimal places}
\end{aligned}
$$

MATCHED PROBLEM 1

Solve $35^{1-2x} = 7$ for x to four decimal places.

EXAMPLE 2 Compound Interest

A certain amount of money P (principal) is invested at an annual rate r compounded annually. The amount of money A in the account after n years, assuming no withdrawals, is given by

$$A = P\left(1 + \frac{r}{m}\right)^n = P(1 + r)^n \qquad m = 1 \text{ for annual compounding.}$$

How many years to the nearest year will it take the money to double if it is invested at 6% compounded annually?

SOLUTION

To find the doubling time, we replace A in $A = P(1.06)^n$ with $2P$ and solve for n.

$$\begin{aligned} 2P &= P(1.06)^n && \text{Divide both sides by } P. \\ 2 &= 1.06^n && \text{Take the common or natural log of both sides.} \\ \log 2 &= \log 1.06^n && \text{Note how log properties are used to get } n \text{ out of the exponent position.} \\ \log 2 &= n \log 1.06 && \text{Divide both sides by } \log 1.06. \\ n &= \frac{\log 2}{\log 1.06} && \text{Calculate to the nearest year.} \\ &= 12 \text{ years} \end{aligned}$$

MATCHED PROBLEM 2

Repeat Example 2, changing the interest rate to 9% compounded annually.

EXAMPLE 3 Atmospheric Pressure

The atmospheric pressure P, in pounds per square inch, at x miles above sea level is given approximately by

$$P = 14.7e^{-0.21x}$$

At what height will the atmospheric pressure be half the sea-level pressure? Compute the answer to two significant digits.

SOLUTION

Sea-level pressure is the pressure at $x = 0$. Thus,

$$P = 14.7e^0 = 14.7$$

One-half of sea-level pressure is $14.7/2 = 7.35$. Now our problem is to find x so that $P = 7.35$; that is, we solve $7.35 = 14.7e^{-0.21x}$ for x:

$$\begin{aligned} 7.35 &= 14.7e^{-0.21x} && \text{Divide both sides by 14.7 to simplify.} \\ 0.5 &= e^{-0.21x} && \text{Because the base is e, take the natural log of both sides.} \\ \ln 0.5 &= \ln e^{-0.21x} && \ln e^a = a \\ \ln 0.5 &= -0.21x && \text{Divide both sides by } -0.21. \\ x &= \frac{\ln 0.5}{-0.21} && \text{Calculate to two significant digits.} \\ &= 3.3 \text{ miles} \end{aligned}$$

MATCHED PROBLEM 3

Using the formula in Example 3, find the altitude in miles so that the atmospheric pressure will be one-eighth that at sea level. Compute the answer to two significant digits.

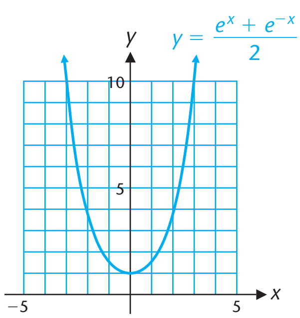


Figure 1
Catenary.

The graph of

$$y = \frac{e^x + e^{-x}}{2} \tag{1}$$

is a curve called a **catenary** (Fig. 1). A uniform cable suspended between two fixed points is a physical example of such a curve.

EXAMPLE 4 Solving an Exponential Equation

Given equation (1), find x for $y = 2.5$. Compute the answer to four decimal places.

SOLUTION

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} && \text{Let } y = 2.5. \\ 2.5 &= \frac{e^x + e^{-x}}{2} && \text{Multiply both sides by 2.} \\ 5 &= e^x + e^{-x} && \text{Multiply both sides by } e^x. \\ 5e^x &= e^{2x} + 1 && \text{Subtract } 5e^x \text{ from both sides.} \\ e^{2x} - 5e^x + 1 &= 0 && \text{This is a quadratic in } e^x. \end{aligned}$$

Let $u = e^x$, then

$$\begin{aligned}
u^2 - 5u + 1 &= 0 && \text{Use the quadratic formula.}\\
u &= \frac{5 \pm \sqrt{25 - 4(1)(1)}}{2} && \text{Simplify.}\\
&= \frac{5 \pm \sqrt{21}}{2} && \text{Replace } u \text{ with } e^x \text{ and solve for } x.\\
e^x &= \frac{5 \pm \sqrt{21}}{2} && \text{Take the natural log of both sides (both values on the right are positive).}\\
\ln e^x &= \ln \frac{5 \pm \sqrt{21}}{2} && \log_b b^x = x.\\
x &= \ln \frac{5 \pm \sqrt{21}}{2} && \text{Exact solutions}\\
&= -1.5668,\ 1.5668 && \text{Rounded to four decimal places.}
\end{aligned}$$

Note that the method produces exact solutions, an important consideration in certain calculus applications (see Problems 61–64 of Exercises 5-5).

MATCHED PROBLEM 4

Given $y = (e^x - e^{-x})/2$, find x for $y = 1.5$. Compute the answer to three decimal places.

Technology Connections

Let $y = e^{2x} + 3e^x + e^{-x}$

(A) Try to find x when $y = 7$ using the method of Example 4. Explain the difficulty that arises.

(B) Use a graphing utility to find x when $y = 7$.

Logarithmic Equations

We now illustrate the solution of several types of logarithmic equations.

EXAMPLE 5 Solving a Logarithmic Equation

Solve $\log(x + 3) + \log x = 1$, and check.

SOLUTION

First use properties of logarithms to express the left side as a single logarithm, then convert to exponential form and solve for x.

$$\begin{aligned} \log(x + 3) + \log x &= 1 && \text{Combine left side using } \log M + \log N = \log MN. \\ \log[x(x + 3)] &= 1 && \text{Change to equivalent exponential form.} \\ x(x + 3) &= 10^1 && \text{Write in } ax^2 + bx + c = 0 \text{ form and solve.} \\ x^2 + 3x - 10 &= 0 && \text{Factor.} \\ (x + 5)(x - 2) &= 0 && \text{If } ab = 0\text{, then } a = 0 \text{ or } b = 0. \\ x &= -5, 2 \end{aligned}$$

CHECK

$x = -5$: $\log(-5 + 3) + \log(-5)$ is not defined because the domain of the log function is $(0, \infty)$.

$$\begin{aligned} x = 2: \log(2 + 3) + \log 2 &= \log 5 + \log 2 \\ &= \log(5 \cdot 2) = \log 10 \overset{\checkmark}{=} 1 \end{aligned}$$

Thus, the only solution to the original equation is $x = 2$. Remember, answers should be checked in the original equation to see whether any should be discarded.

MATCHED PROBLEM 5

Solve $\log(x - 15) = 2 - \log x$, and check.

EXAMPLE 6 Solving a Logarithmic Equation

Solve $(\ln x)^2 = \ln x^2$.

SOLUTION

There are no logarithmic properties for simplifying $(\ln x)^2$. However, we can simplify $\ln x^2$, obtaining an equation involving $\ln x$ and $(\ln x)^2$.

$$\begin{aligned} (\ln x)^2 &= \ln x^2 && \ln M^p = p \ln M \\ (\ln x)^2 &= 2 \ln x && \text{This is a quadratic equation in } \ln x. \text{ Move all nonzero terms to the left.} \\ (\ln x)^2 - 2 \ln x &= 0 && \text{Factor.} \\ (\ln x)(\ln x - 2) &= 0 && \text{If } ab = 0\text{, then } a = 0 \text{ or } b = 0. \end{aligned}$$

$$\begin{aligned} \ln x &= 0 & \quad\text{or}\quad & & \ln x - 2 &= 0 \\ x &= e^0 & & & \ln x &= 2 \\ &= 1 & & & x &= e^2 \end{aligned}$$

Checking that both $x = 1$ and $x = e^2$ are solutions to the original equation is left to you.

MATCHED PROBLEM 6

Solve $\log x^2 = (\log x)^2$.

››› CAUTION ›››

Note that

$$(\log_b x)^2 \neq \log_b x^2 \qquad \begin{matrix}(\log_b x)^2 = (\log_b x)(\log_b x) \\ \log_b x^2 = 2 \log_b x\end{matrix}$$

EXAMPLE 7

Earthquake Intensity

Recall from Section 5-4 that the magnitude of an earthquake on the Richter scale is given by

$$M = \frac{2}{3} \log \frac{E}{E_0}$$

Solve for E in terms of the other symbols.

SOLUTION

$$\begin{aligned} M &= \frac{2}{3} \log \frac{E}{E_0} && \text{Multiply both sides by } \tfrac{3}{2} \text{ and switch sides.} \\ \log \frac{E}{E_0} &= \frac{3M}{2} && \text{Change to exponential form.} \\ \frac{E}{E_0} &= 10^{3M/2} && \text{Multiply both sides by } E_0. \\ E &= E_0 10^{3M/2} \end{aligned}$$

MATCHED PROBLEM 7

Solve the rocket equation from Section 5-4 for W_b in terms of the other symbols:

$$v = c \ln \frac{W_t}{W_b}$$

ANSWERS TO MATCHED PROBLEMS

1. $x = 0.2263$ **2.** More than double in 9 years, but not quite double in 8 years
3. 9.9 miles **4.** $x = 1.195$ **5.** $x = 20$ **6.** $x = 1{,}100$ **7.** $W_b = W_t e^{-v/c}$

5-5 Exercises

In Problems 1–12, solve to three significant digits.

1. $10^{-x} = 0.0347$

2. $10^{x} = 14.3$

3. $10^{3x+1} = 92$

4. $10^{5x-2} = 348$

5. $e^{x} = 3.65$

6. $e^{-x} = 0.0142$

7. $e^{2x-1} + 68 = 207$

8. $13 + e^{3x+5} = 23$

9. $5^{x} = 4^{2x+1}$

10. $3^{x} = 2^{x-1}$

11. $2^{3}2^{-x} = 0.426$

12. $3^{4}3^{-x} = 0.089$

In Problems 13–18, solve exactly.

13. $\log 5 + \log x = 2$

14. $\log x - \log 8 = 1$

15. $\log x + \log (x - 3) = 1$

16. $\log (x - 9) + \log 100x = 3$

17. $\log (x + 1) - \log (x - 1) = 1$

18. $\log (2x + 1) = 1 + \log (x - 2)$

In Problems 19–26, solve to three significant digits.

19. $2 = 1.05^{x}$

20. $3 = 1.06^{x}$

21. $e^{-1.4x} + 5 = 0$

22. $e^{0.32x} + 0.47 = 0$

23. $123 = 500e^{-0.12x}$

24. $438 = 200e^{0.25x}$

25. $e^{-x^2} = 0.23$

26. $e^{x^2} = 125$

In Problems 27–42, solve exactly.

27. $\log (5 - 2x) = \log (3x + 1)$

28. $\log (x + 3) = \log (6 + 4x)$

29. $\log x - \log 5 = \log 2 - \log (x - 3)$

30. $\log (6x + 5) - \log 3 = \log 2 - \log x$

31. $\ln x = \ln (2x - 1) - \ln (x - 2)$

32. $\ln (x + 1) = \ln (3x + 1) - \ln x$

33. $\log (2x + 1) = 1 - \log (x - 1)$

34. $1 - \log (x - 2) = \log (3x + 1)$

35. $\ln (x + 1) = \ln (3x + 3)$

36. $1 + \ln (x + 1) = \ln (x - 1)$

37. $(\ln x)^3 = \ln x^4$

38. $(\log x)^3 = \log x^4$

39. $\ln (\ln x) = 1$

40. $\log (\log x) = 1$

41. $x^{\log x} = 100x$

42. $3^{\log x} = 3x$

In Problems 43–52, find the x and y intercepts of each function without graphing.

43. $f(x) = 4 - e^{x/2}$

44. $f(x) = e^{3x} - 5$

45. $g(x) = 2 + \ln (x + 1)$

46. $g(x) = -8 - \ln (x - 3)$

47. $h(x) = 2^{x+3} + 1$

48. $h(x) = 3^{4-x} + 2$

49. $k(x) = \log_{1/3} x - 5$

50. $k(x) = \log_{1/2} x + 7$

51. $m(x) = \log (x + 8) - \log (2x + 1)$

52. $m(x) = 1 - 2 \log x + \log (x - 1)$

Solve Problems 53–60 for the indicated variable in terms of the remaining symbols. Use the natural log for solving exponential equations.

53. $A = Pe^{rt}$ for r (finance)

54. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ for t (finance)

55. $D = 10 \log \frac{I}{I_0}$ for I (sound)

56. $t = \frac{-1}{k}(\ln A - \ln A_0)$ for A (decay)

57. $M = 6 - 2.5 \log \frac{I}{I_0}$ for I (astronomy)

58. $L = 8.8 + 5.1 \log D$ for D (astronomy)

59. $I = \frac{E}{R}(1 - e^{-Rt/L})$ for t (circuitry)

60. $S = R\frac{(1 + i)^n - 1}{i}$ for n (annuity)

The following combinations of exponential functions define four of six ***hyperbolic functions,*** *an important class of functions in calculus and higher mathematics. Solve Problems 61–64 for x in terms of y. The results are used to define* ***inverse hyperbolic functions,*** *another important class of functions in calculus and higher mathematics.*

61. $y = \dfrac{e^x + e^{-x}}{2}$

62. $y = \dfrac{e^x - e^{-x}}{2}$

63. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

64. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

In Problems 65–76, use a graphing utility to approximate to two decimal places any solutions of the equation in the interval $0 \le x \le 1$. None of these equations can be solved exactly using any step-by-step algebraic process.

65. $2^{-x} - 2x = 0$

66. $3^{-x} - 3x = 0$

67. $x3^x - 1 = 0$

68. $x2^x - 1 = 0$

69. $e^{-x} - x = 0$

70. $xe^{2x} - 1 = 0$

71. $xe^x - 2 = 0$

72. $e^{-x} - 2x = 0$

73. $\ln x + 2x = 0$

74. $\ln x + x^2 = 0$

75. $\ln x + e^x = 0$

76. $\ln x + x = 0$

APPLICATIONS

77. COMPOUND INTEREST How many years, to the nearest year, will it take a sum of money to double if it is invested at 7% compounded annually?

78. COMPOUND INTEREST How many years, to the nearest year, will it take money to quadruple if it is invested at 6% compounded annually?

79. COMPOUND INTEREST At what annual rate compounded continuously will $1,000 have to be invested to amount to $2,500 in 10 years? Compute the answer to three significant digits.

80. COMPOUND INTEREST How many years will it take $5,000 to amount to $8,000 if it is invested at an annual rate of 9% compounded continuously? Compute the answer to three significant digits.

81. WORLD POPULATION A mathematical model for world population growth over short periods is given by

$$P = P_0 e^{rt}$$

where P is the population after t years, P_0 is the population at $t = 0$, and the population is assumed to grow continuously at the annual rate r. How many years, to the nearest year, will it take the world population to double if it grows continuously at an annual rate of 1.14%?

★**82. WORLD POPULATION** Refer to Problem 81. Starting with a world population of 6.5 billion people and assuming that the population grows continuously at an annual rate of 1.14%, how many years, to the nearest year, will it be before there is only 1 square yard of land per person? Earth contains approximately 1.7×10^{14} square yards of land.

★**83. ARCHAEOLOGY—CARBON-14 DATING** As long as a plant or animal is alive, carbon-14 is maintained in a constant amount in its tissues. Once dead, however, the plant or animal ceases taking in carbon, and carbon-14 diminishes by radioactive decay according to the equation

$$A = A_0 e^{-0.000124t}$$

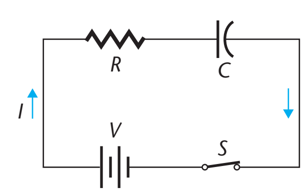
where A is the amount after t years and A_0 is the amount when $t = 0$. Estimate the age of a skull uncovered in an archaeological site if 10% of the original amount of carbon-14 is still present. Compute the answer to three significant digits.

★**84. ARCHAEOLOGY—CARBON-14 DATING** Refer to Problem 83. What is the half-life of carbon-14? That is, how long will it take for half of a sample of carbon-14 to decay? Compute the answer to three significant digits.

★**85. PHOTOGRAPHY** An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time.* For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

How many seconds will it take the capacitor to reach a charge of 0.0007 coulomb? Compute the answer to three significant digits.



★**86. ADVERTISING** A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million possible viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

How many days, to the nearest day, will the advertising campaign have to last so that 80% of the possible viewers will be aware of the product?

★★**87. NEWTON'S LAW OF COOLING** This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in a refrigerator at 40°F to cool before a dinner party. After an hour the temperature of the wine is found to be 61.5°F. Find the constant k, to two decimal places, and the time, to one decimal place, it will take the wine to cool from 72 to 50°F.

★**88. MARINE BIOLOGY** Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity is reduced according to the exponential function

$$I = I_0 e^{-kd}$$

where I is the intensity d feet below the surface and I_0 is the intensity at the surface. The constant k is called the *coefficient of extinction.* At Crystal Lake in Wisconsin it was found that half the surface light remained at a depth of 14.3 feet. Find k, and find the depth of the photic zone. Compute answers to three significant digits.

CHAPTER 5 Review

5-1 Exponential Functions

The equation $f(x) = b^x$, $b > 0$, $b \neq 1$, defines an **exponential function** with **base *b*.** The **domain** of f is $(-\infty, \infty)$ and the **range** is $(0, \infty)$. The **graph** of f is a continuous curve that has no sharp corners; passes through (0, 1); lies above the x axis, which is a horizontal asymptote; increases as x increases if $b > 1$; decreases as x increases if $b < 1$; and intersects any horizontal line at most once. The function f is one-to-one and has an inverse. We have the following **exponential function properties:**

1. $a^x a^y = a^{x+y}$ $\quad (a^x)^y = a^{xy}$ $\quad (ab)^x = a^x b^x$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

2. $a^x = a^y$ if and only if $x = y$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$.

As x approaches ∞, the expression $[1 + (1/x)]^x$ approaches the irrational number $e \approx 2.718\,281\,828\,459$. The function $f(x) = e^x$ is called the **exponential function with base *e*.** The growth of money in an account paying **compound interest** is described by $A = P(1 + r/m)^n$, where P is the **principal,** r is the annual **rate,** m is the number of compounding periods in 1 year, and A is the **amount** in the account after n compounding periods.

If the account pays **continuous compound interest,** the amount A in the account after t years is given by $A = Pe^{rt}$.

5-2 Exponential Models

Exponential functions are used to model various types of growth (see Table 3 on p. 483):

1. **Population growth** can be modeled by using the **doubling time growth model** $P = P_0 2^{t/d}$, where P is population at time t. P_0 is the population at time $t = 0$, and d is the **doubling time**—the time it takes for the population to double. Another model of population growth, $y = ce^{kt}$, where c and k are positive constants, uses the exponential function with base e; k is the **relative growth rate.**

2. **Radioactive decay** can be modeled by using the **half-life decay model** $A = A_0(\frac{1}{2})^{t/h} = A_0 2^{-t/h}$, where A is the amount at time t, A_0 is the amount at time $t = 0$, and h is the **half-life**—the time it takes for half the material to decay. Another model of radioactive decay, $y = ce^{-kt}$, where c and k are positive constants, uses the exponential function with base e.

3. **Limited growth**—the growth of a company or proficiency at learning a skill, for example—can often be modeled by the equation $y = c(1 - e^{-kt})$, where c and k are positive constants.

4. **Logistic growth**—the spread of an epidemic or sales of a new product, for example—can often be modeled by the equation $y = M/(1 + ce^{-kt})$ where c, k, and M are positive constants.

5-3 Logarithmic Functions

The **logarithmic function with base b** is defined to be the inverse of the exponential function with base b and is denoted by $y = \log_b x$. Thus, $y = \log_b x$ if and only if $x = b^y$, $b > 0$, $b \neq 1$. The **domain** of a logarithmic function is $(0, \infty)$ and the **range** is $(-\infty, \infty)$. The graph of a logarithmic function is a continuous curve that always passes through the point $(1, 0)$ and has the y axis as a vertical asymptote. We have the following **properties of logarithmic functions:**

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ if and only if $M = N$

Logarithms to the base 10 are called **common logarithms** and are denoted by $\log x$. Logarithms to the base e are called **natural logarithms** and are denoted by $\ln x$. Thus, $\log x = y$ is equivalent to $x = 10^y$, and $\ln x = y$ is equivalent to $x = e^y$.

The **change-of-base formula,** $\log_b N = (\log_a N)/(\log_a b)$, relates logarithms to two different bases and can be used, along with a calculator, to evaluate logarithms to bases other than e or 10.

5-4 Logarithmic Models

The following applications involve logarithmic functions:

1. The **decibel** is defined by $D = 10 \log (I/I_0)$, where D is the **decibel level** of the sound I is the **intensity** of the sound, and $I_0 = 10^{-12}$ watts per square meter is a standardized sound level.
2. The **magnitude** M of an earthquake on the **Richter scale** is given by $M = \frac{2}{3} \log (E/E_0)$, where E is the energy released by the earthquake and $E_0 = 10^{4.40}$ joules is a standardized energy level.
3. The **velocity** v of a rocket at burnout is given by the **rocket equation** $v = c \ln (W_t/W_b)$, where c is the exhaust velocity, W_t is the takeoff weight, and W_b is the burnout weight.

Logarithmic regression is used to fit a function of the form $y = a + b \ln x$ to a set of data points.

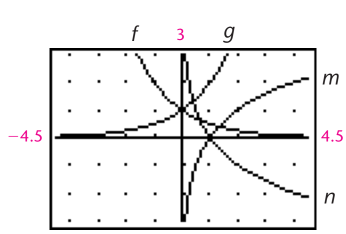
5-5 Exponential and Logarithmic Equations

Various techniques for solving **exponential equations,** such as $2^{3x-2} = 5$, and **logarithmic equations,** such as $\log (x + 3) + \log x = 1$, are illustrated by examples.

CHAPTER 5 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Match each equation with the graph of f, g, m, or n in the figure.
(A) $y = \log_2 x$ (B) $y = 0.5^x$
(C) $y = \log_{0.5} x$ (D) $y = 2^x$



2. Write in logarithmic form using base 10: $m = 10^n$.
3. Write in logarithmic form using base e: $x = e^y$.

Write Problems 4 and 5 in exponential form.

4. $\log x = y$ 5. $\ln y = x$

In Problems 6 and 7, simplify.

6. $\dfrac{7^{x+2}}{7^{2-x}}$ 7. $\left(\dfrac{e^x}{e^{-x}}\right)^x$

Solve Problems 8–10 for x exactly. Do not use a calculator or table.

8. $\log_2 x = 3$ 9. $\log_x 25 = 2$ 10. $\log_3 27 = x$

Solve Problems 11–14 for x to three significant digits.

11. $10^x = 17.5$

12. $e^x = 143{,}000$

13. $\ln x = -0.015\,73$

14. $\log x = 2.013$

Evaluate Problems 15–18 to four significant digits using a calculator.

15. $\ln \pi$

16. $\log(-e)$

17. $\pi^{\ln 2}$

18. $\dfrac{e^{\pi} + e^{-\pi}}{2}$

Solve Problems 19–29 for x exactly. Do not use a calculator or table.

19. $\ln(2x - 1) = \ln(x + 3)$

20. $\log(x^2 - 3) = 2\log(x - 1)$

21. $e^{x^2-3} = e^{2x}$

22. $4^{x-1} = 2^{1-x}$

23. $2x^2e^{-x} = 18e^{-x}$

24. $\log_{1/4} 16 = x$

25. $\log_x 9 = -2$

26. $\log_{16} x = \frac{3}{2}$

27. $\log_x e^5 = 5$

28. $10^{\log_{10} x} = 33$

29. $\ln x = 0$

Solve Problems 30–39 for x to three significant digits.

30. $x = 2(10^{1.32})$

31. $x = \log_5 23$

32. $\ln x = -3.218$

33. $x = \log(2.156 \times 10^{-7})$

34. $x = \dfrac{\ln 4}{\ln 2.31}$

35. $25 = 5(2^x)$

36. $4{,}000 = 2{,}500(e^{0.12x})$

37. $0.01 = e^{-0.05x}$

38. $5^{2x-3} = 7.08$

39. $\dfrac{e^x - e^{-x}}{2} = 1$

Solve Problems 40–45 for x exactly. Do not use a calculator.

40. $\log 3x^2 - \log 9x = 2$

41. $\log x - \log 3 = \log 4 - \log(x + 4)$

42. $\ln(x + 3) - \ln x = 2\ln 2$

43. $\ln(2x + 1) - \ln(x - 1) = \ln x$

44. $(\log x)^3 = \log x^9$

45. $\ln(\log x) = 1$

In Problems 46 and 47, simplify.

46. $(e^x + 1)(e^{-x} - 1) - e^x(e^{-x} - 1)$

47. $(e^x + e^{-x})(e^x - e^{-x}) - (e^x - e^{-x})^2$

In Problems 48–51, use transformations to explain how the graph of g is related to the graph of the given logarithmic function f. Determine whether g is increasing or decreasing, find its domain and any asymptotes, and sketch the graph of g.

48. $g(x) = 3 - \frac{1}{3}2^x;\ f(x) = 2^x$

49. $g(x) = 2e^x - 4;\ f(x) = e^x$

50. $g(x) = -2 + \log_4 x;\ f(x) = \log_4 x$

51. $g(x) = 1 + 2\log_{1/3} x;\ f(x) = \log_{1/3} x$

52. If the graph of $y = e^x$ is reflected in the line $y = x$, the graph of the function $y = \ln x$ is obtained. Discuss the functions that are obtained by reflecting the graph of $y = e^x$ in the x axis and the y axis.

53. (A) Explain why the equation $e^{-x/3} = 4\ln(x + 1)$ has exactly one solution.

(B) Find the solution of the equation to three decimal places.

54. Approximate all real zeros of $f(x) = 4 - x^2 + \ln x$ to three decimal places.

55. Find the coordinates of the points of intersection of $f(x) = 10^{x-3}$ and $g(x) = 8\log x$ to three decimal places.

Solve Problems 56–59 for the indicated variable in terms of the remaining symbols.

56. $D = 10\log\dfrac{I}{I_0}$ for I (sound intensity)

57. $y = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for x (probability)

58. $x = -\dfrac{1}{k}\ln\dfrac{I}{I_0}$ for I (X-ray intensity)

59. $r = P\dfrac{i}{1 - (1 + i)^{-n}}$ for n (finance)

60. Write $\ln y = -5i + \ln c$ in an exponential form free of logarithms; then solve for y in terms of the remaining symbols.

61. For $f = \{(x, y) \mid y = \log_2 x\}$, graph f and f^{-1} on the same coordinate system. What are the domains and ranges for f and f^{-1}?

62. Explain why 1 cannot be used as a logarithmic base.

63. Prove that $\log_b(M/N) = \log_b M - \log_b N$.

APPLICATIONS

64. POPULATION GROWTH Many countries have a population growth rate of 3% (or more) per year. At this rate, how many years will it take a population to double? Use the annual compounding growth model $P = P_0(1 + r)^t$. Compute the answer to three significant digits.

65. **POPULATION GROWTH** Repeat Problem 64 using the continuous compounding growth model $P = P_0e^{rt}$.

66. **CARBON 14-DATING** How many years will it take for carbon-14 to diminish to 1% of the original amount after the death of a plant or animal? Use the formula $A = A_0e^{-0.000124t}$. Compute the answer to three significant digits.

*__67.__ **MEDICINE** One leukemic cell injected into a healthy mouse will divide into two cells in about $\frac{1}{2}$ day. At the end of the day these two cells will divide into four. This doubling continues until 1 billion cells are formed; then the animal dies with leukemic cells in every part of the body.
(A) Write an equation that will give the number N of leukemic cells at the end of t days.
(B) When, to the nearest day, will the mouse die?

68. **MONEY GROWTH** Assume $1 had been invested at an annual rate 3% compounded continuously at the birth of Christ. What would be the value of the account in the year 2000? Compute the answer to two significant digits.

69. **PRESENT VALUE** Solving $A = Pe^{rt}$ for P, we obtain $P = Ae^{-rt}$, which is the **present value** of the amount A due in t years if money is invested at a rate r compounded continuously.
(A) Graph $P = 1{,}000(e^{-0.08t})$, $0 \le t \le 30$.
(B) What does it appear that P tends to as t tends to infinity? [*Conclusion:* The longer the time until the amount A is due, the smaller its present value, as we would expect.]

70. **EARTHQUAKES** The 1971 San Fernando, California, earthquake released 1.99×10^{14} joules of energy. Compute its magnitude on the Richter scale using the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules. Compute the answer to one decimal place.

71. **EARTHQUAKES** Refer to Problem 70. If the 1906 San Francisco earthquake had a magnitude of 8.3 on the Richter scale, how much energy was released? Compute the answer to three significant digits.

*__72.__ **SOUND** If the intensity of a sound from one source is 100,000 times that of another, how much more is the decibel level of the louder sound than the softer one? Use the formula $D = 10\log(I/I_0)$.

__73.__ **MARINE BIOLOGY The intensity of light entering water is reduced according to the exponential function

$$I = I_0e^{-kd}$$

where I is the intensity d feet below the surface, I_0 is the intensity at the surface, and k is the coefficient of extinction. Measurements in the Sargasso Sea in the West Indies have indicated that half the surface light reaches a depth of 73.6 feet. Find k, and find the depth at which 1% of the surface light remains. Compute answers to three significant digits.

*__74.__ **WILDLIFE MANAGEMENT** A lake formed by a newly constructed dam is stocked with 1,000 fish. Their population is expected to increase according to the logistic curve

$$N = \frac{30}{1 + 29e^{-1.35x}}$$

where N is the number of fish, in thousands, expected after t years. The lake will be open to fishing when the number of fish reaches 20,000. How many years, to the nearest year, will this take?

Problems 75 and 76 require a graphing calculator or a computer that can calculate exponential and logarithmic regression models for a given data set.

75. **MEDICARE** The annual expenditures for Medicare (in billions of dollars) by the U.S. government for selected years since 1980 are shown in Table 1 (Bureau of the Census). Let x represent years since 1975.
(A) Find an exponential regression model of the form $y = ab^x$ for these data. Estimate (to the nearest billion) the total expenditures in 2010.
(B) When (to the nearest year) will the total expenditures reach $500 billion?

Table 1 Medicare Expenditures

Year	Billion $
1980	37
1985	72
1990	111
1995	181
2000	225

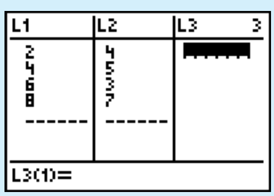
Source: U.S. Census Bureau.

76. (A) Refer to Problem 75. Find a logarithmic regression model of the form $y = a + b\ln x$ for the data in Table 1. Use the model to estimate the total Medicare expenditures in 2010.
(B) Which regression model, exponential or logarithmic, better fits the data? Justify your answer.

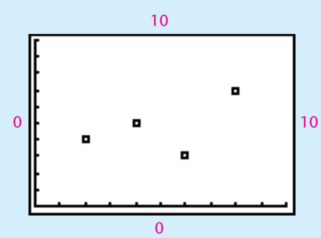
CHAPTER 5

››› GROUP ACTIVITY Comparing Regression Models

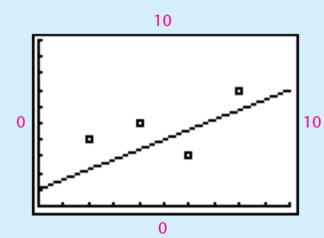
We have used polynomial, exponential, and logarithmic regression models to fit curves to data sets. How can we determine which equation provides the best fit for a given set of data? There are two principal ways to select models. The first is to use information about the type of data to help make a choice. For example, we expect the weight of a fish to be related to the cube of its length. And we expect most populations to grow exponentially, at least over the short term. The second method for choosing among equations involves developing a measure of how closely an equation fits a given data set. This is best introduced through an example. Consider the data set in Figure 1, where L1 represents the x coordinates and L2 represents the y coordinates. The graph of this data set is shown in Figure 2. Suppose we arbitrarily choose the equation $y_1 = 0.6x + 1$ to model these data (Fig. 3).



› **Figure 1**

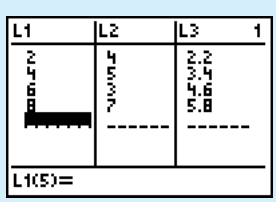


› **Figure 2**

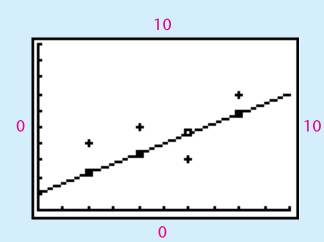


› **Figure 3** $y_1 = 0.6x + 1$.

To measure how well the graph of y_1 fits these data, we examine the difference between the y coordinates in the data set and the corresponding y coordinates on the graph of y_1 (L3 in Figs. 4 and 5).



› **Figure 4**



› **Figure 5** Here + is L2 and □ is L3.

Each of these differences is called a **residual.** Note that three of the residuals are positive and one is negative (three of the points lie above the line, one lies below). The most commonly accepted measure of the fit provided by a given model is the **sum of the squares of the residuals (SSR).** When squared, each residual (whether positive or negative or zero) makes a nonnegative contribution to the SSR.

$$\begin{aligned} \text{SSR} &= (4 - 2.2)^2 + (5 - 3.4)^2 + (3 - 4.6)^2 + (7 - 5.8)^2 \\ &= (1.8)^2 + (1.6)^2 + (-1.6)^2 + (1.2)^2 \\ &= 9.8 \end{aligned}$$

Technology Connections

Calculating the SSR for a data set and model involves only basic arithmetic. But a graphing calculator makes the computation especially easy (see Fig. 6)

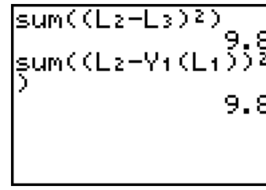


Figure 6 Two ways to calculate SSR.

(A) A linear regression model for the data in Figure 1 is given by

$$y_2 = 0.35x + 3$$

Compute the SSR for the data and y_2, and compare it to the one we computed for y_1.

It turns out that among all possible linear polynomials, *the linear regression model minimizes the sum of the squares of the residuals.* For this reason, the linear regression model is often called the **least-squares line.** A similar statement can be made for polynomials of any fixed degree. That is, the quadratic regression model minimizes the SSR over all quadratic polynomials, the cubic regression model minimizes the SSR over all cubic polynomials, and so on. The same statement cannot be made for exponential or logarithmic regression models. Nevertheless, the SSR can still be used to compare exponential, logarithmic, and polynomial models.

(B) Find the exponential and logarithmic regression models for the data in Figure 1, compute their SSRs, and compare with the linear model.

(C) National annual advertising expenditures for selected years since 1950 are shown in Table 1 where x is years since 1950 and y is total expenditures in billions of dollars. Which regression model would fit this data best: a quadratic model, a cubic model, or an exponential model? Use the SSRs to support your choice.

Table 1 Annual Advertising Expenditures, 1950–2000

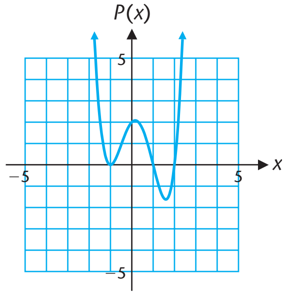
x (years)	0	10	20	30	40	50
y (billion \$)	5.7	12.0	19.6	53.6	128.6	247.5

Source: U.S. Bureau of the Census.

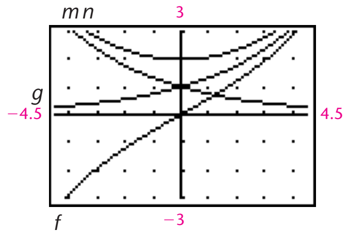
CHAPTERS 4–5 Cumulative Review Exercises

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Let $P(x)$ be the polynomial whose graph is shown in the figure.
(A) Assuming that $P(x)$ has integer zeros and leading coefficient 1, find the lowest-degree equation that could produce this graph.
(B) Describe the left and right behavior of $P(x)$.



2. Match each equation with the graph of f, g, m, or n in the figure.
(A) $y = (\frac{3}{4})^x$ (B) $y = (\frac{4}{3})^x$
(C) $y = (\frac{3}{4})^x + (\frac{4}{3})^x$ (D) $y = (\frac{4}{3})^x - (\frac{3}{4})^x$



3. For $P(x) = 3x^3 + 5x^2 - 18x - 3$ and $D(x) = x + 3$, use synthetic division to divide $P(x)$ by $D(x)$, and write the answer in the form $P(x) = D(x)Q(x) + R$.

4. Let $P(x) = 2(x + 2)(x - 3)(x - 5)$. What are the zeros of $P(x)$?

5. Let $P(x) = 4x^3 - 5x^2 - 3x - 1$. How do you know that $P(x)$ has at least one real zero between 1 and 2?

6. Let $P(x) = x^3 + x^2 - 10x + 8$. Find all rational zeros for $P(x)$.

7. Solve for x.
(A) $y = 10^x$ (B) $y = \ln x$

8. Simplify.
(A) $(2e^x)^3$ (B) $\dfrac{e^{3x}}{e^{-2x}}$

9. Solve for x exactly. Do not use a calculator or a table.
(A) $\log_3 x = 2$
(B) $\log_3 81 = x$
(C) $\log_x 4 = -2$

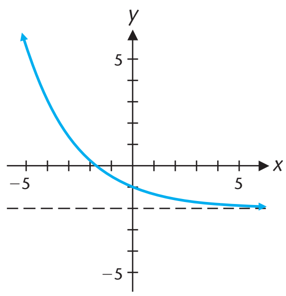
10. Solve for x to three significant digits.
(A) $10^x = 2.35$ (B) $e^x = 87{,}500$
(C) $\log x = -1.25$ (D) $\ln x = 2.75$

In Problems 11 and 12, translate each statement into an equation using k as the constant of proportionality.

11. E varies directly as p and inversely as the cube of x.

12. F is jointly proportional to q_1 and q_2 and inversely proportional to the square of r.

13. Explain why the graph in the figure is not the graph of a polynomial function.



14. Explain why the graph in the figure is not the graph of a rational function.

15. The function f subtracts the square root of the domain element from three times the natural log of the domain element. Write an algebraic definition of f.

16. Write a verbal description of the function $f(x) = 100e^{0.5x} - 50$.

17. Let $f(x) = \dfrac{2x + 8}{x + 2}$.
(A) Find the domain and the intercepts for f.
(B) Find the vertical and horizontal asymptotes for f.
(C) Sketch the graph of f. Draw vertical and horizontal asymptotes with dashed lines.

18. Find all zeros of $P(x) = (x^3 + 4x)(x + 4)$, and specify those zeros that are x intercepts.

19. Solve $(x^3 + 4x)(x + 4) \leq 0$.

20. If $P(x) = 2x^3 - 5x^2 + 3x + 2$, find $P(\frac{1}{2})$ using the remainder theorem and synthetic division.

21. Which of the following is a factor of

$$P(x) = x^{25} - x^{20} + x^{15} + x^{10} - x^5 + 1$$

(A) $x - 1$ (B) $x + 1$

22. Let $P(x) = x^4 - 8x^2 + 3$.
(A) Graph $P(x)$ and describe the graph verbally, including the number of x intercepts, the number of turning points, and the left and right behavior.
(B) Approximate the largest x intercept to two decimal places.

23. Let $P(x) = x^5 - 8x^4 + 17x^3 + 2x^2 - 20x - 8$.
(A) Approximate the zeros of $P(x)$ to two decimal places and state the multiplicity of each zero.
(B) Can any of these zeros be approximated with the bisection method? A maximum routine? A minimum routine? Explain.

24. Let $P(x) = x^4 + 2x^3 - 20x^2 - 30$.
(A) Find the smallest positive and largest negative integers that, by Theorem 1 in Section 4-2, are upper and lower bounds, respectively, for the real zeros of $P(x)$.
(B) If $(k, k + 1)$, k an integer, is the interval containing the largest real zero of $P(x)$, determine how many additional intervals are required in the bisection method to approximate this zero to one decimal place.
(C) Approximate the real zeros of $P(x)$ to two decimal places.

25. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = 4x^3 - 20x^2 + 29x - 15$.

26. Final all zeros (rational, irrational, and imaginary) exactly for $P(x) = x^4 + 5x^3 + x^2 - 15x - 12$, and factor $P(x)$ into linear factors.

Solve Problems 27–36 for x exactly. Do not use a calculator or a table.

27. $2^{x^2} = 4^{x+4}$ **28.** $2x^2e^{-x} + xe^{-x} = e^{-x}$

29. $e^{\ln x} = 2.5$ **30.** $\log_x 10^4 = 4$

31. $\log_9 x = -\frac{3}{2}$

32. $\ln (x + 4) - \ln (x - 4) = 2 \ln 3$

33. $\ln (2x^2 + 2) = 2 \ln (2x - 4)$

34. $\log x + \log (x + 15) = 2$

35. $\log (\ln x) = -1$ **36.** $4 (\ln x)^2 = \ln x^2$

Solve Problems 37–41 for x to three significant digits.

37. $x = \log_3 41$ **38.** $\ln x = 1.45$

39. $4(2^x) = 20$ **40.** $10e^{-0.5x} = 1.6$

41. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{2}$

42. G is directly proportional to the square of x. If $G = 10$ when $x = 5$, find G when $x = 7$.

43. H varies inversely as the cube of r. If $H = 162$ when $r = 2$, find H when $r = 3$.

In Problems 44–50, find the domain, range, and the equations of any horizontal or vertical asymptotes.

44. $f(x) = 3 + 2^x$

45. $f(x) = 2 - \log_3 (x - 1)$

46. $f(x) = 5 - 4x^3$

47. $f(x) = 3 + 2x^4$

48. $f(x) = \dfrac{5}{x + 3}$

49. $f(x) = 20e^{-x} - 15$

50. $f(x) = 8 + \ln (x + 2)$

51. If the graph of $y = \ln x$ is reflected in the line $y = x$, the graph of the function $y = e^x$ is obtained. Discuss the functions that are obtained by reflecting the graph of $y = \ln x$ in the x axis and in the y axis.

52. (A) Explain why the equation $e^{-x} = \ln x$ has exactly one solution.
(B) Approximate the solution of the equation to two decimal places.

In Problems 53 and 54, factor each polynomial in two ways:
(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)
(B) As a product of linear factors with complex coefficients

53. $P(x) = x^4 + 9x^2 + 18$

54. $P(x) = x^4 - 23x^2 - 50$

55. Graph f and indicate any horizontal, vertical, or oblique asymptotes with dashed lines:

$$f(x) = \frac{x^2 + 4x + 8}{x + 2}$$

56. Let $P(x) = x^4 - 28x^3 + 262x^2 - 922x + 1.083$. Approximate (to two decimal places) the x intercepts and the local extrema.

57. Find a polynomial of lowest degree with leading coefficient 1 that has zeros -1 (multiplicity 2), 0 (multiplicity 3), $3 + 5i$, and $3 - 5i$. Leave the answer in factored form. What is the degree of the polynomial?

58. If $P(x)$ is a fourth-degree polynomial with integer coefficients and if i is a zero of $P(x)$, can $P(x)$ have any irrational zeros? Explain.

59. Let $P(x) = x^4 + 9x^3 - 500x^2 + 20{,}000$.

(A) Find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that, by Theorem 1 in Section 4-2, are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) Approximate the real zeros of $P(x)$ to two decimal places.

60. Find all zeros (rational, irrational, and imaginary) exactly for

$$P(x) = x^5 - 4x^4 + 3x^3 + 10x^2 - 10x - 12$$

and factor $P(x)$ into linear factors.

61. Find rational roots exactly and irrational roots to two decimal places for

$$P(x) = x^5 + 4x^4 + x^3 - 11x^2 - 8x + 4$$

62. Give an example of a rational function $f(x)$ that satisfies the following conditions: the real zeros of f are 5 and 8; $x = 1$ is the only vertical asymptote; and the line $y = 3$ is a horizontal asymptote.

63. Use natural logarithms to solve for n.

$$A = P\frac{(1 + i)^n - 1}{i}$$

64. Solve $\ln y = 5x + \ln A$ for y. Express the answer in a form that is free of logarithms.

65. Solve for x.

$$y = \frac{e^x - 2e^{-x}}{2}$$

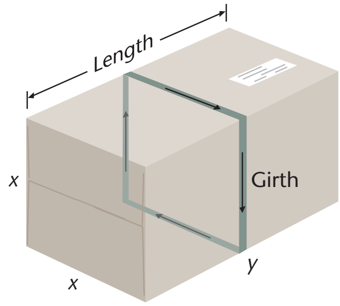
66. Solve $\dfrac{x^3 - x}{x^3 - 8} \geq 0$.

67. Solve (to three decimal places)

$$\frac{4x}{x^2 - 1} < 3$$

APPLICATIONS

68. SHIPPING A mailing service provides customers with rectangular shipping containers. The length plus the girth of one of these containers is 10 feet (see the figure). If the end of the container is square and the volume is 8 cubic feet, find the side length of the end. Find solutions exactly; round irrational solutions to two decimal places.



69. GEOMETRY The diagonal of a rectangle is 2 feet longer than one of the sides, and the area of the rectangle is 6 square feet. Find the dimensions of the rectangle to two decimal places.

70. POPULATION GROWTH If the Democratic Republic of the Congo has a population of about 60 million people and a doubling time of 23 years, find the population in

(A) 5 years (B) 30 years

Compute answers to three significant digits.

71. COMPOUND INTEREST How long will it take money invested in an account earning 7% compounded annually to double? Use the annual compounding growth model $P = P_0(1 + r)^t$, and compute the answer to three significant digits.

72. COMPOUND INTEREST Repeat Problem 71 using the continuous compound interest model $P = P_0e^{rt}$.

73. EARTHQUAKES If the 1906 and 1989 San Francisco earthquakes registered 8.3 and 7.1, respectively, on the Richter scale, how many times more powerful was the 1906 earthquake than the 1989 earthquake? Use the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules, and compute the answer to one decimal place.

74. SOUND If the decibel level at a rock concert is 88, find the intensity of the sound at the concert. Use the formula $D = 10\log(I/I_0)$, where $I_0 = 10^{-12}$ watts per square meter, and compute the answer to two significant digits.

75. ASTRONOMY The square of the time t required for a planet to make one orbit around the sun varies directly as the cube of its mean (average) distance d from the sun. Write the equation of variation, using k as the constant of variation.

★**76.** PHYSICS Atoms and molecules that make up the air constantly fly about like microscopic missiles. The velocity v of a particular particle at a fixed temperature varies inversely as the square root of its molecular weight w. If an oxygen molecule in air at room temperature has an average velocity of 0.3 mile/second, what will be the average velocity of a hydrogen molecule, given that the hydrogen molecule is one-sixteenth as heavy as the oxygen molecule?

Problems 77 and 78 require a graphing calculator or a computer that can calculate linear, quadratic, cubic, and exponential regression models for a given data set.

77. Table 1 shows the life expectancy (in years) at birth for residents of the United States from 1970 to 2000. Let x represent years since 1970. Use the indicated regression model to estimate the life expectancy (to the nearest tenth of a year) for a U.S. resident born in 2010.

(A) Linear regression (B) Quadratic regression
(C) Cubic regression (D) Exponential regression

Table 1

Year	Life Expectancy
1970	70.8
1975	72.6
1980	73.7
1985	74.7
1990	75.4
1995	75.9
2000	77.0

Source: U.S Census Bureau.

78. Refer to Problem 77. The Census Bureau projected the life expectancy for a U.S. resident born in 2010 to be 77.6 years. Which of the models in Problem 77 is closest to the Census Bureau projection?