* Evaluate the Improper Integrals:

1 $\int_0^\infty \frac{dx}{x^2+1}$

Sol'n:

$$I = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \left[\tan^{-1}(x)\right]_0^t$$

$$= \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(0)\right]$$

$$= \tan^{-1}(\infty) - 0 = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}.$$

2 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol'n:

$$I = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \left[\sin^{-1}(x)\right]_0^t$$

$$= \lim_{t\to 1^-} \left[\sin^{-1}(t) - \sin^{-1}(0)\right]$$

$$= \sin^{-1}(1) - 0 = \frac{\pi}{2}.$$

③ $\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx$

Soln:-

$$I = \int_{-\infty}^{0} \frac{2x}{(x^2+1)^2} dx + \int_{0}^{\infty} \frac{2x}{(x^2+1)^2} dx$$

$$= \lim_{t \to -\infty} \int_{t}^{0} \frac{2x}{(x^2+1)^2} dx + \lim_{s \to \infty} \int_{0}^{s} \frac{2x\,dx}{(x^2+1)^2}$$

$$I_1 = \lim_{t \to -\infty} \int_{t}^{0} \frac{2x}{(x^2+1)^2} dx$$

$$= \lim_{t \to -\infty} \int_{t}^{0} (x^2+1)^{-2} \cdot 2x\,dx$$

$$= \lim_{t \to -\infty} \left[\frac{-1}{(x^2+1)}\right]_{t}^{0} = \lim_{t \to -\infty} \left[-1 + \frac{1}{t^2+1}\right]$$

$$= -1 . \longrightarrow (1)$$

$$I_2 = \lim_{s \to \infty} \int_0^s \frac{2x}{(x^2+1)^2} \, dx$$

$$= \lim_{s \to \infty} \int_0^s (x^2+1)^{-2} \cdot 2x \, dx$$

$$= \lim_{s \to \infty} \left[\frac{-1}{x^2+1} \right]_0^s = \lim_{s \to \infty} \left[\frac{-1}{s^2+1} + 1 \right]$$

$$= 1. \longrightarrow (2)$$

Where $I = I_1 + I_2 = 1 - 1 = 0.$

[illegible]

$$\boxed{4} \int_0^2 \frac{s+1}{\sqrt{4-s^2}} \, ds$$

Sol:

$$I = \lim_{t \to 2^-} \int_0^t \frac{s+1}{\sqrt{4-s^2}} \, ds$$

$$= \lim_{t \to 2^-} \left[\int_0^t \frac{s}{\sqrt{4-s^2}} \, ds + \int_0^t \frac{1}{\sqrt{4-s^2}} \, ds \right]$$

$$= \lim_{t \to 2^-} \left[\frac{-1}{2} \int_0^t (4 - s^2)^{-1/2} \cdot (-2s)\, ds + \frac{1}{2} \int_0^t \frac{1}{\sqrt{1 - \frac{s^2}{4}}}\, ds \right]$$

$$= \lim_{t \to 2^-} \left[-\frac{1}{\cancel{2}} \cdot \cancel{2} (4 - s^2)^{1/2} + \cancel{\tfrac{1}{2}} \sin^{-1}\left(\frac{s}{2}\right) \right]_0^t$$

$$= \lim_{t \to 2^-} \left[-(4 - t^2)^{1/2} + 2 + \cancel{\tfrac{1}{2}} \sin^{-1}\left(\frac{t}{2}\right) - \cancel{\tfrac{1}{2}} \cancel{\sin^{-1}(0)} \right]$$

صفر

$$= 2 + \frac{\pi}{2} = \frac{4 + \pi}{2}.$$

$\boxed{5}$ $\int_{-\infty}^{\infty} 2x^{-x^2}\, dx$

Sol:

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[6] $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}\,dx$

Soln: By using matlab Programme to solve it

or use Fresnel itegral. (it's advanced Ex:!!)

In any Case.

Let $\sqrt{x} = u \Rightarrow du = \frac{dx}{2\sqrt{x}}$

$x: 0 \longrightarrow \pi$

$u: 0 \longrightarrow \sqrt{\pi}$

$$\therefore I = \int_0^{\pi} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\sqrt{\pi}} \sin u^2\,du$$

By using Fresnel integral. we have

$I = \sqrt{2\pi}\, S(\sqrt{2})$, where $S(x)$ is the Fresnel Power series.

* Testing For Convergence :-

1 $\displaystyle\int_0^{\pi/2} \tan\theta \, d\theta$

Soln:-

$$I = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \tan\theta \, d\theta = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \frac{\sin\theta}{\cos\theta} \, d\theta$$

$$= -\lim_{t \to \frac{\pi}{2}^-} \Big[\ln|\cos\theta|\Big]_0^t = -\lim_{t \to \frac{\pi}{2}} \Big(\ln|\cos t| - \underset{\ln(o)}{\cancel{0}}\Big)$$

$$= -\ln\left|\cos\left(\tfrac{\pi}{2}\right)\right| = +\infty \quad (\underline{\text{divergence}})$$

2 $\displaystyle\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

Soln:-

$$I = \lim_{t \to \infty} \int_2^t \frac{dv}{\sqrt{v-1}} = \lim_{t \to \infty} \int_2^t (v-1)^{-1/2} \, dv$$

$$= \lim_{t \to \infty} \Big[2\sqrt{v-1}\Big]_2^t = \lim_{t \to \infty} \Big[2\sqrt{t-1} - 2\Big]$$

$$= 2\lim_{t \to \infty} \left[\sqrt{t-1} - 1 * \frac{\sqrt{t-1} + 1}{\sqrt{t-1} + 1}\right]$$

$$= 2 \lim_{t\to\infty} \frac{t-1-1}{\sqrt{t-1}+1} = 2 \lim_{t\to\infty} \frac{t-2}{\sqrt{t-1}+1}$$

$$= +\infty \text{ (divergence).}$$

$$\boxed{B} \int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$$

Soln: Let $f(x) = \frac{\sqrt{x+1}}{x^2}$ and $g(x) = \frac{\sqrt{x}+1}{x^2}$

Where $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{x+1}}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{\sqrt{x}+1}$

$$= 1.$$

And $f(x) \le g(x) \Rightarrow \frac{\sqrt{x+1}}{x^2} \le \frac{\sqrt{x}+1}{x^2}$

Now, we calculate the integral

$$I = \int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{\sqrt{x}+1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^{t} \frac{\sqrt{x}+1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} (x^{-3/2} + x^{-2})\,dx$$

$$= \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}} + \frac{1}{x}\right]_1^t$$

$$= \lim_{t \to \infty} \left[\frac{-2}{\sqrt{t}} - \frac{1}{t} + 2 + 1 \right]$$

$$= \lim_{t \to \infty} \left[3 - \left(\frac{2t + \sqrt{t}}{t\sqrt{t}} \right) \right]$$

$$= \lim_{t \to \infty} 3 - \lim_{t \to \infty} \left[\frac{2t + t^{1/2}}{t^{3/2}} \right]$$

$$= 3 - 0 = 3.$$

i.e. $\int_1^{\infty} g(x)\,dx$ is Convergent $\Rightarrow \int_1^{\infty} f(x)\,dx$ is Convergent.

$\boxed{4}$ $\int_{\pi}^{\infty} \frac{2 + \cos x}{x}\,dx$

Soln:- Let $f(x) = \frac{2 + \cos x}{x}$ and $g(x) = \frac{3}{x}$

Where

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{2 + \cos x}{\cancel{x}} \cdot \frac{\cancel{x}}{3}$$

Since $|\cos x| \leq 1$

$$\therefore \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

And $f(x) \leq g(x) \Rightarrow \frac{2+\cos x}{x} \leq \frac{3}{x}$

Now, we calculate the integral

$$I = \int_{\pi}^{\infty} g(x)\,dx = \int_{\pi}^{\infty} \frac{3}{x}\,dx$$

$$= \lim_{t \to \infty} \int_{\pi}^{t} \frac{3}{x}\,dx = 3 \lim_{t \to \infty} (\ln t - \ln \pi)$$

$$= +\infty \text{ (divergent)}$$

In this case, ~~No since~~ This doesn't say anything about the smaller function ($f(x)$).

Now, we will look for another function

Let $h(x) = \frac{1}{x}$, where $f(x) \geq h(x)$

we will calculate the integral

$$I^* = \int_{\pi}^{\infty} h(x)\,dx = \int_{\pi}^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{t \to \infty} \int_{\pi}^{t} \frac{1}{x}\,dx = \lim_{t \to \infty} [\ln x]_{\pi}^{t}$$

$= \lim_{t \to \infty} \left(\ln(t) - \ln(\pi) \right) = +\infty$ (divergent)

Then $\int_{\pi}^{\infty} \frac{2 + \cos x}{x} \, dx$ is divergence

$\ast$

$\boxed{5}$ $\int_{-\infty}^{\infty} \frac{dx}{e^{x} + e^{-x}}$

Soln:

$\therefore I = \underbrace{\int_{-\infty}^{0} \frac{dx}{e^{x} + e^{-x}}}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{dx}{e^{x} + e^{-x}}}_{I_2}$

Where $\cosh x = \frac{e^{x} + e^{-x}}{2}$

$\therefore I_1 = \frac{1}{2} \int_{-\infty}^{0} \frac{2}{e^{x} + e^{-x}} \, dx = \frac{1}{2} \int_{-\infty}^{0} \operatorname{sech} x \, dx$

$= \frac{1}{2} \lim_{t \to -\infty} \int_{t}^{0} \operatorname{sech} x \, dx$

$$= \frac{1}{2} \lim_{t \to -\infty} \left[\tan^{-1}\left(\sinh(x) \right) \right]_t^0$$

$$= \frac{1}{2} \lim_{t \to -\infty} \left[\tan^{-1}\left(\frac{e^{x} - e^{-x}}{2} \right) \right]_t^0$$

$$= \frac{1}{2} \lim_{t \to -\infty} \left[\tan^{-1}(0) - \tan^{-1}\left(\frac{e^{t} - e^{-t}}{2} \right) \right]$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \longrightarrow (1)$$

$$I_2 = \frac{1}{2} \lim_{s \to \infty} \int_0^{\infty} \operatorname{sech} x \, dx$$

$$= \frac{1}{2} \lim_{s \to \infty} \left[\tan^{-1}\left(\frac{e^{x} - e^{-x}}{2} \right) \right]_0^s$$

$$= \frac{1}{2} \lim_{s \to \infty} \left[\tan^{-1}\left(\frac{e^{s} - e^{-s}}{2} \right) - \tan^{-1}(0) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{\pi}{2} \right] = 0 \longrightarrow (2)$$

$$\therefore I = I_1 + I_2 = \frac{\pi}{2} \longrightarrow (\text{Convergent})$$

$\boxed{6}$ $\int_0^2 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

<u>Soln:</u>

$$I = \lim_{t \to 0^+} \int_t^2 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= -2 \lim_{t \to 0^+} \int_t^2 -\frac{e^{-\sqrt{x}}}{2\sqrt{x}}\,dx$$

$$= -2 \lim_{t \to 0^+} \left[e^{-\sqrt{x}}\right]_t^2$$

$$= -2 \lim_{t \to 0^+} \left(e^{-\sqrt{2}} + e^{-\sqrt{t}}\right)$$

$$= -2\left(e^{-\sqrt{2}} - 1\right) \rightsquigarrow (\underline{\text{Convergent}})$$

ملاحظة:

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